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Time-delay Elimination for the Analysis and Control for a Class of Retarded Dynamical Systems

Thesis

to partially fulfil the requirements needed to obtain the degree of

Doctor in Science

Presents:

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Resumen de la tesis que presenta Eduardo García Ramírez como requisito parcial para la obtención del grado de Doctor en Ciencias en Electrónica y Telecomunicaciones.

Eliminación de retardos en el tiempo para el análisis y control de una clase de sistemas dinámicos con retardos

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Los resultados de este estudio cubren temas de linealización mediante cambio de coordenadas y la aproximación a sistemas lineales libres de retardo. Primero, se presentan resultados generales que demuestran que el uso de la aproximación por series de Taylor truncada a partir de cierto orden en una ecuación diferencial con retardo lleva a una ecuación diferencial ordinaria inestable. Considerando este problema, se introducen aproximaciones libres de retardo que lidian con esta inconsistencia para la ecuación diferencial lineal de primer orden con retardo puro y para una clase de ecuaciones hereditarias de segundo orden. Estos resultados permiten el uso de teoría de control para sistemas sin retardo en el diseño de leyes de control y observadores de estados. Además, se reportan contribuciones en la linealización de sistemas con retardos por medio de herramientas de control geométrico y algebraico. En años recientes, una extensión para sistemas hereditarios del corchete de Lie ha sido usada en la solución de diversos problemas de linealización. Se presenta in algoritmo eficiente para el cálculo iterativo de esta operación geométrica. Además, herramientas algebraicas son usadas para establecer condiciones constructivas para la equivalencia de sistemas no lineales con retardo, por medio de cambios de coordenadas, a sistemas que son lineales excepto por una parte no lineal que depende sólo de la entrada y la salida. Más aún, se dan condiciones para la bicausalidad de esta transformación. Dicha representación es usada en el diseño de un observador tipo Luenberger que permite el cálculo de los valores del estado en el pasado o en el presente. Todos juntos, estos resultados permiten encontrar una representación lineal sin retardos para una clase de sistemas no lineales observables con retardos en el tiempo. Esto se hace linealizando el sistema mediante un cambio de coordenadas y luego, de ser necesario, usando la aproximación por series de Taylor para encontrar una representación libre de retardos. Esta estrategía permite el uso de teoría de control sin retardos.

Palabras Clave: **Sistemas con retardo, sistemas no lineales, eliminación de retardos, observadores no lineales con retardos, aproximación por series de Taylor**

Abstract of the thesis presented by Eduardo García Ramírez as a partial requirement to obtain the Doctor in Science degree in Electronics and Telecomunications.

Time-delay Elimination for the Analysis and Control for a Class of Retarded Dynamical Systems

Abstract approved by:

Dr. Luis Alejandro Márquez Martínez
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The aim of this work is the investigation of techniques for the simplification of nonlinear time-delay systems. The outcomes of this study cover results of linearization throughout coordinates transformation and approximation to delay-free linear systems. First, general results are given that show that the use of Taylor series approximations truncated at a given order on time-delay linear systems leads to unstable ordinary differential equations. Considering this problem, Taylor-approximated delay-free differential equations that deal with this inconsistency are presented for a pure-delay first-order differential equation, and for a class of linear second order hereditary system. This result permits to design control laws or state observers using linear delay-free control theory. Contributions are reported on linearization using geometric and algebraic tools for time-delay systems. In the recent years an extension of the Lie bracket, valid for hereditary systems, has been used in the solution of several linearization problems. An efficient algorithm, developed in this work, that computes this geometrical operation in an iterative manner is presented. Algebraic tools are used to give constructive conditions for the equivalence of a time-delay nonlinear system, via a change of coordinates, to a time delay system which is linear observable except for a nonlinear part that depends on the input and the output only. Conditions are also given to ensure the bicausality of this transformation. This representation is used to design a Luenberger-type observer that allows the computation of the present or the past values of the state variables. Altogether, these results permit to find a linear delay-free representation for a certain class of observable time-delay nonlinear systems. This is done by linearizing the time-delay nonlinear system via a change of coordinates and then, if necessary, using a Taylor series approximation to find a delay-free representation. This strategy allows the use of delay-free control theory.

Keywords: **Time-delay, nonlinear systems, delay elimination, time-delay nonlinear observers, Taylor series approximation**

Dedication

To my parents,

Miguel Ángel and María Élida.

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Notation

$\mathbb{Z}^+$	Set of non-negative integers.
$\mathbf{x}(t)$	Vector of instantaneous system state.
$\mathbf{x}_e$	Vector of instantaneous and delayed system state $(\mathbf{x}^T(t) \;\; \mathbf{x}^T(t-1) \; \cdots \; \mathbf{x}^T(t-s'))^T$ for a context-defined $s' \in \mathbb{Z}^+$.
$\mathbf{x}_{[p,s]}$	Denotes the list $(\mathbf{x}(t+p), \ldots, \mathbf{x}(t-s))$ for $p, s \geq 0$. $\mathbf{z}_{[p,s]}$, and $\mathbf{u}_{[p,s]}$ are defined in a similar way. $\mathbf{x}_{[s]}$ denotes $\mathbf{x}_{[0,s]}$.
$\bar{\mathbf{u}}$	Represents $(u^T(t), \dot{u}^T(t), \ldots, (u^{(n-1)}(t))^T)^T$. $\bar{\mathbf{y}}$ is defined in a similar way.
$\chi(\mathbf{x}_{[l]})\big\|_{\mathbf{x}_{[l]}(-j)}$	Defines $\chi(\mathbf{x}(t-j), \mathbf{x}(t-j-1), \ldots, \mathbf{x}(t-j-l))$.
$\mathcal{K}$	Field of meromorphic functions of the symbols $\{\mathbf{x}(t-i), u(t-i), \ldots, u^{(k)}(t-i), i \in \mathbb{Z}, k \in \mathbb{Z}^+\}$
$\mathcal{K}(\delta]$	Non-commutative Euclidean (left) ring of polynomials with coefficients over $\mathcal{K}$, and indeterminate δ.
$\mathbb{R}[\delta]$	Ring of polynomials in δ with coefficients in $\mathbb{R}$
$deg(\cdot)$	$deg(\cdot)$ the polynomial degree in δ of its argument
$\mathcal{E}$	Defined as $\mathcal{E} = span_{\mathcal{K}}\{dx(t-i), du(t-i), \ldots, du^{(k)}(t-i), i \in \mathbb{Z}, k \in \mathbb{N}\}$.
d	Symbol defined as the differential operator, that maps elements from $\mathcal{K}$ to $\mathcal{E}$.
$\mathcal{M}$	Left-module over the ring $\mathcal{K}(\delta]$: $\mathcal{M} = span_{\mathcal{K}(\delta]}\{d\xi \mid \xi \in \mathcal{K}\}$.
Ω	Represents the $span_{\mathcal{K}(\delta]}\{\omega_i(\mathbf{x}, \delta), \;\; i = 1, \ldots, p\}$ which is the module spanned over $\mathcal{K}(\delta]$ by the row vectors $\omega_1(\mathbf{x}, \delta)$, $\omega_2(\mathbf{x}, \delta)$, $\ldots$, $\omega_p(\mathbf{x}, \delta) \in \mathcal{K}^n(\delta]$.
$diag(\cdot)$	Represents the diagonal matrix constructed by the elements in the argument.
$addcol(\cdot)$	Operator that concatenate columns of the elements in the argument.

Chapter 1. Introduction.

1.1 Time-Delay Systems Background.

A system whose evolution depends on its own behavior occurred in the past, is known as time-delay system. The period of time that takes to a certain event of the past to affect a given system is called time-delay. Time-delay systems are equally known as hereditary, memory, retarded, aftereffect, or time-lag systems. This kind of systems is studied using the theory of retarded differential equations (RDE) which are a particular case of the so-called functional differential equations, also known as differential equations with deviating argument. Sources of the aftereffect can be due to material transportation, computational delays, retarded measurements of the variable, signal transmissions, the hereditary physical nature of some components of the system, among others. Since the early years of the twentieth century, the scientific community had already reported several phenomena that presented different behaviors depending on the past history. This was reported in the well-known congress article Picard (1908) where the author stated.

"Mechanics, as we mentioned earlier, was more or less explicitly considered as a non-hereditary principle. We still accept this principle, at least as a first approximation, to study inanimate nature, even though various phenomena show that the current state keeps a record of past states."

The speech given by Picard was considered by V. Volterra who, in his lessons Volterra (1913), treated the concept of hereditary mechanics and traced applications of this kind of systems to Boltzmann (1878). He also speculated about the possibility that this kind of systems was also considered by Leibniz.

As pointed out in Hale (2006), where several important publications are enlisted, retarded systems gained popularity at the end of the first half of the twentieth century, particularly in the soviet union, since aftereffects were observed in engineering problems. The contributions in this field, due to the consideration of the memory effect, have been growing until this days. At the beginning of the current century, Richard (2003) presented several advances in the study of this kind of systems concerning stability, structural properties, and several approaches and tools. The attention given to the time-lag effect has been growing since it has been observed in several fields (engineering, chemistry, physics, life science, economics, etc.). A large number of publications can be found covering advances in the topic as described in Erneux (2009). Stability, chaos, and periodicity of single, and multi-species are presented in Kuang (1993). In Smith (2011) the delay effect on dynamical models of virus transmission and bacteria growing is analyzed.

One way to deal with the analysis and control problem is to find if a given system is equivalent, or may be approximated to a linear form. This approach permits to take advantage of the well-known properties from the linear system theory. In the case of the delay-free framework, two different techniques are commonly used for the linearization of nonlinear dynamical systems, Taylor series expansion of the state variables, and coordinate transformations. The Taylor series expansion approach is a well-known strategy that consists in approximating a function by a power series neglecting the higher-order terms. The coordinate transformation, or change of coordinates methods, are mainly studied in control theory by two different mathematical approaches, geometric, and algebraic. The geometric control theory was introduced in the decade of 1970 to study linear systems (see for example Wonham (1979)). Later on, the concepts created for linear systems were taken to the nonlinear context, taking advantage of the mathematical theory of differential geometry. Important results in this field can be found in Isidori (1995). More recently, the algebraic control theory takes advantage of differential algebraic tools to solve, among other, linearization problems (see Conte *et al.* (2007)). In the time-delay framework, Taylor series expansion, and geometric and algebraic theory have had different levels of success in the solution of the problem. The use of Taylor series expansion to substitute the delayed variable for delay elimination presents inconsistencies in the stability parametric regions, see some examples in Insperger (2015). Nevertheless, aftereffect elimination problems, for commensurable time-delay systems, have been successfully solved in the linear case (Gárate-García *et al.*, 2011), and in the nonlinear case using algebraic and geometric approaches (see *e.g.* Márquez-Martínez *et al.* (2002) for the algebraic approach, and (Califano *et al.*, 2010, 2011a; Califano and Moog, 2011; Califano *et al.*, 2011b; Califano and Moog, 2012b,a; Califano *et al.*, 2013b; Califano and Moog, 2014) for the geometric approach). The algebraic approach takes advantage of the properties given by the polynomial ring defined using the so-called delay operator. Meanwhile, several meaningful results reached by the geometric approach have been possible thanks to the development of an extension, for time-delay systems, of the Lie bracket, cornerstone operation on the differential geometry based delay-free results.

The main objective of this thesis is to find conditions to allow the use of control and analysis strategies already defined in the classical control theory framework. To achieve this goal, the following strategy is proposed: first, find if the system is equivalent to a linear delay system, if the resultant representation is a hereditary system, use the Taylor series expansion approach to eliminate the delays. Second, use delay-free techniques for the study, and control law design, and finally, go back to the original nonlinear coordinates. According to the strategy proposed, the following particular objectives are formulated:

- Find constructive conditions for the equivalence of a nonlinear time-delay system with a linear system, up to input-output injection, through a change of coordinates.
- For a linear time-delay system, using a Taylor series expansion techniques, find a delay-free approximated linear dynamical system.
- Contribute in the computational simplification of the, recently extended, geometrical tools for time-delay systems.

The main contributions of this work are structured as follows. General conditions are given under which the Taylor series approximation of a family of linear delay differential equations is tractable. Analytical results state the degree of truncation for which the substitution of the delayed variable, on a class of linear delay system, by its Taylor series approximation leads to an unstable ordinary differential equation (ODE). Moreover, Taylor series approximations that deal with the inconsistencies reported in the literature are delineated for the pure-delay first-order linear differential equation, and for a family of second-order linear differential equations. Furthermore, contributions to the change of coordinates techniques for delay elimination where made in the algebraic, and the geometric frameworks. Algebraic tools are used to find a solution for the problem of linearization via input-output injection, by means of causal and bicausal transformations. Contributions are made in the geometric theory by the characterization of properties of the extended Lie bracket that permit to construct an efficient algorithm for the effective computation of such a tool. Finally, delay-free results for the computation of the relative degree of nonlinear time-delay systems are given.

The diagram in Figure 1 links the contributions given in this thesis. In Figure 1 it is shown that the given results are related with extensions of the order of the dynamics. This relationship will be treated throughout this document.

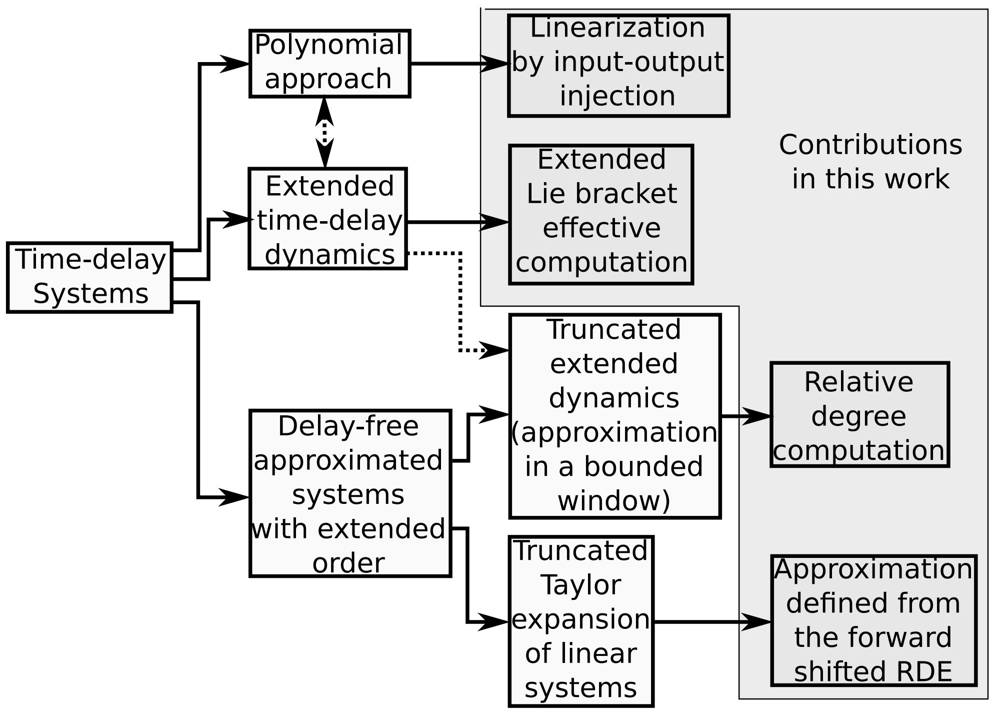


Figure 1: Thesis contributions.

The main contributions of this work can be used to create a framework for the controller or observer design for a nonlinear time-delay system. This is proposed in the following manner. First, check if the nonlinear time-delay system can be linearized by a change of coordinates, by means of elimination of the nonlinear part of the equation. If it is not possible, the tools proposed in this work can not be used to solve the problem. If it is possible, the system can be equivalent to a delay or a delay-free system. If the linear representation of the system is delay dependent, approximate the linear retarded system to a delay-free system. Now use techniques based on delay-free control theory for linear systems. Finally, compute the controller or observer nonlinear form using the inverse change of coordinates. This strategy is summarized in Figure 2.

1.2 Outline.

This document is organized as follows. Chapter 2 introduces the mathematical background, and recalls some results from the literature used in the rest of the thesis. This

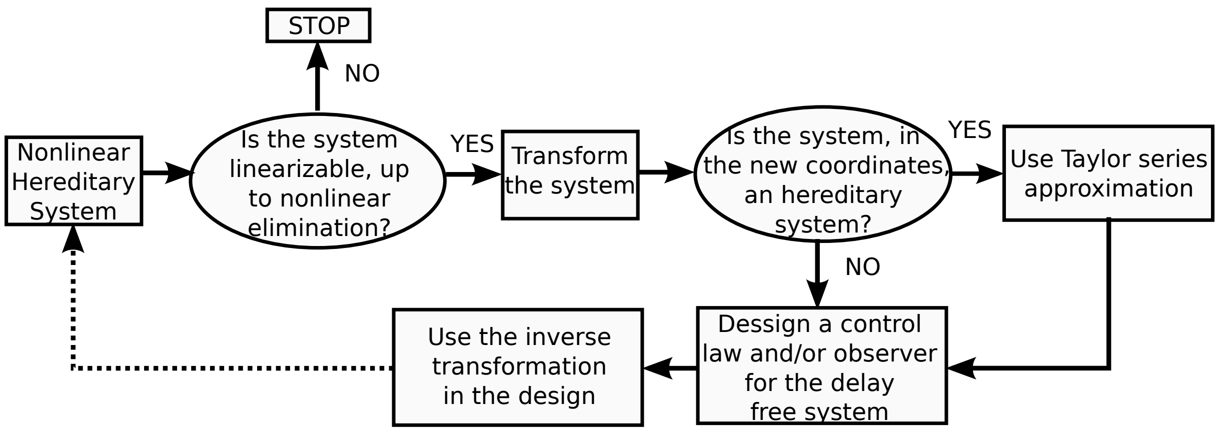


Figure 2: Controller-observer design via delay elimination.

includes the definition of the studied general system, as well as the considered observability definitions. The concept of invertible change of coordinates as a non-causal invertible transformation is recalled from García-Ramírez *et al.* (2016) as a medium to compute coordinates in the past time. Also, we present a review of the algebraic structure, and some remarkable properties established in Márquez-Martínez *et al.* (2000); Xia *et al.* (2002), constructed by the delay operator, as indeterminate, in the polynomial ring with meromorphic functions as coefficients. Finally, in this section, the geometric frame, constructed for systems with aftereffect introduced by Califano *et al.* (2011a), is revised enumerating some achievements in the topic of change of coordinates to canonical forms. Chapter 3 explains how Taylor series expansion can be used to generate a family of approximated equations of a class of linear systems with delays. The importance of this class of equations lies in the presence of several engineering control systems that can be described by this kind of equations (see for example Micheau and Kron (2001), Olgac *et al.* (2004), Liu and Chopra (2012), Singh and Ouyang (2013), Ge and Orosz (2014), Wang *et al.* (2014), Xiang *et al.* (2015), among others). The given results show that the stability regions of the hereditary system and its approximation match locally in the parametric space. Also, it is shown by numerical means that, even if the stability regions match exactly, a certain degree of truncation is necessary for an accurate approximation. Chapter 4 describes the relationship between the well-known step method, for the solution of functional differential equations, with the extended space system proposed by Califano *et al.* (2011a). This approach is illustrative, among other things, to answer how initial conditions of the extended system are settled. Also, taking advantage of the extended system structure, sufficient conditions are presented to find the relative degree of a retarded dynamical system, by means of a finite dimensional system which is a delay-free version of the time-delay extended system, obtained by variable substitution and dimension truncation. At the end of the chapter, an algorithm for the effective computation of the so-called extended Lie bracket for time-delay systems is presented. This algorithm can be used to reduce the number of operations in coordinate transformation problems as the enlisted in Section 2.4. Chapter 5 covers the problem of linearization via input-output injection via invertible, and bicausal transformations. A constructive algorithm, introduced in García-Ramírez *et al.* (2016), is recalled to be used in the solution of the equivalence problem with a time-delay system which is linear up to an injection function that depends on the input and output variables only. Also, results for the effective computation of change of coordinates are presented. The effectiveness of Luenberger-type observer design strategy, based on the canonical form given by the algorithm, is illustrated through numerical solutions of academic examples. Finally, conclusions and perspectives of this work can be found in Chapter 6.

Chapter 2. Mathematical Setting

This chapter deals with the mathematical background used into the rest of the document. It is divided into four main topics. First, the system definition of the general hereditary nonlinear system is presented. Definitions of weak and strong observability are given. Next, Section 2.2 sets basic definitions and properties of the polynomial ring constructed with the delay operator δ as indeterminate over the field of meromorphic functions. Section 2.3 covers the topic of the invertible and bicausal changes of coordinates. A formal definition is given to the concept of invertible change of coordinates, which is an invertible transformation whose inverse is not necessarily causal. On the other hand, bicausality in the transformation ensures the computation of the present state of the system. Finally, Section 2.4 is devoted to the geometrical background.

2.1 Systems Under Study

The commensurable time-delay systems considered in this work can be represented by the equations

$$\begin{aligned} \dot{\mathbf{x}}(t) &= f(\mathbf{x}(t), \ldots, \mathbf{x}(t-s\tau)) + \sum_{j=0}^{s} g_i(\mathbf{x}(t), \ldots, \mathbf{x}(t-s\tau))u(t-j\tau) \\ y(t) &= h(\mathbf{x}(t), \ldots, \mathbf{x}(t-s\tau)), \end{aligned} \tag{1}$$

where the function of initial conditions $\vartheta : [-s\tau, 0] \to \mathbb{R}^n$ is assumed to be continuous. The variables $\mathbf{x}(t) \in \mathbb{R}^n$, $y(t), u(t) \in \mathbb{R}$ denote, respectively, the instantaneous values of the state, output, and input functions. Without loss of generality, after rescaling, the constant base delay τ can be set equal to 1.

The following notation is taken from Califano *et al.* (2011a): $\mathcal{K}$ denotes the field of meromorphic functions of the symbols $\{x(t-i), u(t-i), \ldots, u^{(k)}(t-i), i \in \mathbb{Z}, k \in \mathbb{Z}^+\}$; d is the differential operator that maps elements from $\mathcal{K}$ to $\mathcal{E} = span_{\mathcal{K}}\{dx(t-i), du(t-i), \ldots, du^{(k)}(t-i), i \in \mathbb{Z}, k \in \mathbb{N}\}$. Using the time-shift operator δ as indeterminate, the non-commutative Euclidean (left) ring of polynomials with coefficients over $\mathcal{K}$ is denoted as $\mathcal{K}(\delta]$; $\mathbb{R}[\delta]$ is the ring of polynomials in δ with coefficients in $\mathbb{R}$. $\mathcal{M}$ is defined as the left-module over the ring $\mathcal{K}(\delta]$: $\mathcal{M} = span_{\mathcal{K}(\delta]}\{d\xi \mid \xi \in \mathcal{K}\}$. Let us define for $p, s \geq 0$, by $(\mathbf{x}_{[p,s]}) = (x(t+p), \ldots, x(t-s))$; $(\mathbf{z}_{[p,s]})$, and $(\mathbf{u}_{[p,s]})$, are defined similarly. We will use $x_{[s]}$ for $x_{[0,s]}$. Define $\bar{\mathbf{u}} = (\mathbf{u}^T(t), \dot{\mathbf{u}}^T(t), \ldots, (\mathbf{u}^{(n-1)}(t))^T)^T$, and $\bar{\mathbf{y}}$ is defined in a similar way. For simplicity $y(t)$, $x(t)$, and $u(t)$ will stand for $\mathbf{y}_{[0]}$, $\mathbf{x}_{[0]}$ and $\mathbf{u}_{[0]}$. Consider also $\bar{f}(\mathbf{x}_{[l]}) \mid_{\mathbf{x}_{[l]}(-j)} := \bar{f}(\mathbf{x}(t-j), \mathbf{x}(t-j-1), \ldots, \mathbf{x}(t-j-l))$. Then, it is possible to rewrite equation (1) as

$$\begin{array}{rcl} \dot{\mathbf{x}}(t) & = & f(\mathbf{x}_{[s]}) + \sum_{j=0}^{s} g_j(\mathbf{x}_{[s]})\mathbf{u}(t-j) \\ \mathbf{y}(t) & = & h(\mathbf{x}_{[s]}). \end{array} \tag{2}$$

A fundamental concept for this work is the time-shift operator which is defined as follows:

Definition 1 *Let $\chi(t)$ be a function. The time-delay operator δ acts on $\chi(t)$ in the following way*

$$\delta^{\alpha}\chi(t) = \chi(t-\alpha), \qquad \alpha \in \mathbb{Z}^{+}.$$

The time-shift operator δ acts over the elements of $\mathcal{E}$ as follows: if $a(\cdot), f(\cdot) \in \mathcal{K}$, then

$$\delta(a(t)df(t)) = a(t-1)\delta df(t) = a(t-1)df(t-1). \qquad \diamond$$

The differential-form representation of (2) is given by

$$d\dot{\mathbf{x}}(t) = F(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)d\mathbf{x}(t) + G(\mathbf{x}_{[s]}, \delta)d\mathbf{u}(t), \tag{3}$$

and

$$d\mathbf{y}(t) = H(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)d\mathbf{x}(t), \tag{4}$$

where

$$\begin{array}{rcl} F(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) & = & \sum_{i=0}^{s} \left(\frac{\partial f(\mathbf{x}_{[s]})}{\partial \mathbf{x}(t-i)} + \sum_{j=0}^{s} \mathbf{u}(t-j) \frac{\partial g_j(\mathbf{x}_{[s]})}{\partial \mathbf{x}(t-i)} \right) \delta^i, \\ G(\mathbf{x}_{[s]}, \delta) & = & \sum_{j=0}^{s} g_j(\mathbf{x}_{[s]})\delta^j, \\ H(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) & = & \sum_{i=0}^{s} \frac{\partial h(\mathbf{x}_{[s]})}{\partial \mathbf{x}(t-i)} \delta^i d\mathbf{x}(t). \end{array} \tag{5}$$

Let us consider the definition of the extended Lie derivative for nonlinear time-delay systems expressed in Zheng *et al.* (2011) as

$$L_f h(\mathbf{x}_{[s]}) = L_f^1 h(\mathbf{x}_{[s]}) = \sum_{i=0}^{s} \frac{\partial h(\mathbf{x}_{[s]})}{\partial \mathbf{x}(t-i)} \delta^i f(\mathbf{x}_{[s]}), \tag{6}$$

and $L_f^l h(\mathbf{x}_{[s]})$ is defined as the l-th extended Lie derivative. The observability matrix is established as

$$\mathcal{O}(\mathbf{x}_{[s]}, \delta) d\mathbf{x}_{[0]} = \begin{pmatrix} dh(\mathbf{x}) \\ dL_f h(\mathbf{x}_{[p]}) \\ \vdots \\ dL_f^{n-1} h(\mathbf{x}_{[p]}) \end{pmatrix} = \begin{pmatrix} d\mathbf{y} \\ d\dot{\mathbf{y}} \\ \vdots \\ d\mathbf{y}^{(n-1)} \end{pmatrix}. \tag{7}$$

The characterization of the algebraic observability property is stated by the next definitions (Califano *et al.* (2013b))

Definition 2 *System* (2) *is said to be weakly-observable if the matrix* $\mathcal{O}$ *has full rank around* $\mathbf{x}_{[0]}$*.* ◇

Definition 3 *System* (2) *is said to be strongly observable if the matrix* $\mathcal{O}$ *is unimodular around* $\mathbf{x}_{[0]}$*.* ◇

Let us consider the input-output representation of the form

$$\psi(\mathbf{y}_{[s]}^{(n)}, \mathbf{y}_{[s]}^{(n-1)}, \ldots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-1)}, \ldots, \mathbf{u}_{[s]}) = 0. \tag{8}$$

The next notation is taken from Halás and Anguelova (2013).
Let us define the r dimensional vector $(\nu_1, \ldots, \nu_r) := (\nu_1 \ \ldots \ \nu_r)^T \in \mathcal{K}^r$, and let $\frac{\partial(\nu_1,\ldots,\nu_r)}{\partial x} \in \mathcal{K}^{r \times n}(\delta]$ denote the matrix with entries

$$\left(\frac{\partial(\nu_1, \ldots, \nu_r)}{\partial x} \right)_{j,i} = \sum_{\iota=0}^{s} \frac{\partial \nu_j}{\partial x_i(t-\iota)} \delta^\iota \in \mathcal{K}(\delta]. \tag{9}$$

The observability index $\bar{d}$ is defined as the least nonnegative integer that fulfills

$$rank_{\mathcal{K}(\delta]}\frac{\partial(H,\ldots,H^{(\bar{d}-1)})}{\partial x} = rank_{\mathcal{K}(\delta]}\frac{\partial(H,\ldots,H^{(\bar{d})})}{\partial x}, \tag{10}$$

with $\bar{d} \leq n$.

Definition 4 *Let $\bar{d}$ be the observability index. Then, the input-output equation* (8) *is said to be retarded if*

$$\frac{\partial\psi(\cdot)}{\partial y^{(\bar{d})}(t-i)} = 0$$

for all $i \geq 1$.

Definition 5 *(Halás and Anguelova (2013)) Let $\bar{d}$ be the observability index then, the input-output equation* (8) *is said to be neutral if there exist $i_1 \neq i_2$ such that*

$$\frac{\partial\psi(\cdot)}{\partial y^{(\bar{d})}(t-i)} \neq 0, \qquad for\ i = i_1,\ i = i_2$$

2.2 Algebraical Setting

In this section, concepts, and definitions of the algebraic framework used in this thesis are established. Using the time-shift operator δ as indeterminate, the elements of $\mathcal{K}(\delta]$ may be written as $\alpha(\delta] = \sum_{i=0}^{r_\alpha}\alpha_i(t)\delta^i$, with $\alpha_i \in \mathcal{K}$, and $r_\alpha = deg(\alpha(\delta])$. The operations addition and multiplication for the non-commutative Euclidean (left) ring of polynomials with coefficients over $\mathcal{K}$, $\mathcal{K}(\delta]$, are defined by

$$\alpha(\delta] + \beta(\delta] = \sum_{i=0}^{max\{r_\alpha,r_\beta\}}(\alpha_i(t)+\beta_i(t))\delta^i,$$

and

$$\alpha(\delta]\beta(\delta] = \sum_{i=0}^{r_\alpha}\sum_{j=0}^{r_\beta}\alpha_i(t)\beta_j(t-i)\delta^{i+j}$$

A matrix $M(\mathbf{x}_{[p,s]},\delta) \in \mathcal{K}^{n\times n}(\delta]$ is called unimodular if it has a polynomial inverse. It is called polymodular if there exists a polynomial matrix $M'(\mathbf{x}_{[p,s]},\delta)$ such that

$$M(\mathbf{x}_{[p,s]},\delta)M'(\mathbf{x}_{[p,s]},\delta) = diag(\delta^{k_1},\ldots\delta^{k_n}),$$

for some $k_i \in \mathbb{Z}^+$.

The right-module spanned over $\mathcal{K}(\delta]$ by the column vectors $\mathbf{r}_1(\mathbf{x},\delta),\ldots,\mathbf{r}_p(\mathbf{x},\delta) \in \mathcal{K}^n(\delta]$ is represented $\Delta_\delta = span_{\mathcal{K}(\delta]}\{\mathbf{r}_1(\mathbf{x},\delta),\ldots,\mathbf{r}_p(\mathbf{x},\delta)\}$.

Example 1 *Consider the function $\chi(t) = x^2(t-1)x(t-2) \in \mathcal{K}$, the differential operator d acts on $\chi(t)$ as*

$$d\chi(t) = 2x(t-1)x(t-2)dx(t-1)+x^2(t-1)dx(t-2) = \left(2x(t-1)x(t-2)\delta + x^2(t-1)\delta^2\right)dx(t).$$

In this way, $2x(t-1)x(t-2)dx(t-1) + x^2(t-1)dx(t-2) \in \mathcal{E}$ can be expressed as an element of $\mathcal{M}$ since $\left(2x(t-1)x(t-2)\delta + x^2(t-1)\delta^2\right)dx(t) \in span_{\mathcal{K}(\delta]}\{dx \mid x \in \mathcal{K}\}$, where $2x(t-1)x(t-2)\delta + x^2(t-1)\delta^2 \in \mathcal{K}(\delta]$. ◀

The following properties can be identified for the ring $\mathcal{K}(\delta]$, $\forall a(x_{[s]},\delta), b(x_{[s]},\delta) \in \mathcal{K}(\delta]$ (Márquez-Martínez *et al.*, 2000; Xia *et al.*, 2002).

- Ore Ring: $\exists \alpha(x_{[s]},\delta), \beta(x_{[s]},\delta) \in \mathcal{K}(\delta]$ such that

$$\alpha(x_{[s]},\delta)a(x_{[s]},\delta) = \beta(x_{[s]},\delta)b(x_{[s]},\delta). \tag{11}$$

- Integral ring: $a(x_{[s]},\delta)b(x_{[s]},\delta) \equiv 0$ implies $a(x_{[s]},\delta) \equiv 0$, or $b(x_{[s]},\delta) \equiv 0$.
- Euclides division: $\exists q(x_{[s]},\delta), r(x_{[s]},\delta) \in \mathcal{K}(\delta]$ with $deg(r(x_{[s]},\delta) < deg(q(x_{[s]},\delta))$ such that

$$a(x_{[s]},\delta) = q(x_{[s]},\delta)b(x_{[s]},\delta) + r(x_{[s]},\delta). \tag{12}$$

- Bezout identity: if $a(x_{[s]},\delta)$, and $b(x_{[s]},\delta)$ are prime relative, $\exists q(x_{[s]},\delta), p(x_{[s]},\delta) \in \mathcal{K}(\delta]$

such that

$$p(x_{[s]},\delta)a(x_{[s]},\delta)+q(x_{[s]},\delta)b(x_{[s]},\delta)=1 \tag{13}$$

Considering such properties, it is always possible to find unimodular matrices $P(x_{[s]},\delta)\in\mathcal{K}^{n\times n}(\delta]$, $Q(x_{[s]},\delta)\in\mathcal{K}^{m\times m}(\delta]$ such that a matrix $A(x_{[s]},u_{[s]},\delta)\in\mathcal{K}^{n\times m}(\delta]$ can be expressed as the matrix product

$$A(x_{[s]},u_{[s]},\delta)=P(x_{[s]},\delta)\hat{S}(x_{[s]},u_{[s]},\delta)Q(x_{[s]},u_{[s]},\delta) \tag{14}$$

with $\hat{S}(x_{[s]},u_{[s]},\delta)\in\mathcal{K}^{n\times m}(\delta]$ defined as follows

Definition 6 *Let $\bar{n}\in\mathbb{Z}^{+}$. It is said that a matrix $\hat{S}(x_{[s]},u_{[s]},\delta)\in\mathcal{K}^{n\times m}(\delta]$ is in the Smith pre-form if it has the structure*

$$\hat{S}(x_{[s]},u_{[s]},\delta)=\begin{pmatrix} \alpha_{1,1}(x_{[s]},\delta) & \alpha_{1,2}(x_{[s]},\delta) & \cdots & \alpha_{1,\bar{n}}(x_{[s]},\delta) & 0 & \cdots & 0\\ 0 & \alpha_{2,2}(x_{[s]},\delta) & \cdots & \alpha_{2,\bar{n}}(x_{[s]},\delta) & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0\\ 0 & 0 & \cdots & \alpha_{\bar{n},\bar{n}}(x_{[s]},\delta) & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{15}$$

Let us end this section with the notion of normalized vector, which will be used in the sequel.

Definition 7 *Let $\lambda(x,u,\delta)=[\lambda_1,\cdots,\lambda_n]\in\mathcal{K}^{n}(\delta]$. λ is called a normalized covector if for $\lambda_i=0$, with $i\in[1,j-1]$, $\lambda_j=1$, $j=1,\ldots,n$.*

This means that the first nonzero element of ? is 1.

2.3 Invertible Change of Coordinates

The concept of change of coordinates is of significant importance in automatic control theory on account of the canonical representations that simplifies the analysis of dynamical control systems. In this section, the concepts of invertible change of coordinates and, as a particular case, bicausal change of coordinates are discussed.

Definition 8 *Given the system defined by* (2)*,* $\mathbf{z}(t) = \phi(\mathbf{x}_{[p,s]})$ *is an invertible change of coordinates if there exists a differentiable function* $\bar{\phi}(\mathbf{z}_{[p',s']}) \in \mathcal{K},\ p, s, p', s' \in \mathbb{N}$*, such that* $\bar{\phi}(\mathbf{z}_{[p',s']}) \mid_{\mathbf{z}(t)=\phi(\mathbf{x}_{[p,s]})} = \mathbf{x}(t)$*.* ◇

To the invertible change of coordinates $\mathbf{z}(t) = \phi(\mathbf{x}_{[p,s]})$ we can associate a list of integers $r_i = max\{l \in \mathbb{Z} \mid \frac{\partial \phi_i(\mathbf{x}_{[p,s]})}{\partial \mathbf{x}(t-l)} \equiv 0\}$. Its differential representation can be written as

$$diag\left(\delta^{r_1}, \ldots, \delta^{r_n}\right) d\mathbf{z}(t) = N(\mathbf{x}_{[\bar{s}]}, \delta) d\mathbf{x}(t).$$

For the inverse transformation, the corresponding indices are defined by $k_i = max\{l' \in \mathbb{Z} \mid \frac{\partial \bar{\phi}(\mathbf{z}_{[p',s']})}{\partial \mathbf{z}(t+l')} \equiv 0\}$. The differential representation is

$$d\mathbf{x}_{[0]} = \tilde{N}(\mathbf{z}_{[p',s']}, \delta) \begin{pmatrix} d\mathbf{z}(t+k_1) \\ \vdots \\ d\mathbf{z}(t+k_n) \end{pmatrix}.$$

Consequently

$$diag\left(\delta^{r_1}, \ldots, \delta^{r_n}\right) d\mathbf{z}(t) = N(\mathbf{x}_{[0,\bar{s}]}, \delta)|_{x=\psi(\mathbf{z}_{[p',s']})} \tilde{N}(\mathbf{z}_{[p',s']}, \delta) \begin{pmatrix} d\mathbf{z}(t+k_1) \\ \vdots \\ d\mathbf{z}(t+k_n) \end{pmatrix}.$$

It follows that its

$$diag\left(\delta^{r_1+k_1}, \ldots, \delta^{r_n+k_n}\right) = N(\mathbf{x}_{[\bar{p}]}, \delta)|_{x=\psi(\mathbf{z}_{[p',s']})} \tilde{N}(\mathbf{z}_{[p',s']}, \delta)$$

differential representation is characterized by a polymodular matrix.

Definition 9 *Given the system defined by* (2)*,* $\mathbf{z}(t) = \phi(\mathbf{x}_{[s]})$ *is a bicausal change of coordinates if there exists a differentiable function* $\bar{\phi}(\mathbf{z}_{[s']}) \in \mathcal{K}, \;\; s, s' \in \mathbb{N}$*, such that* $\bar{\phi}(\mathbf{z}_{[s']}) \mid_{\mathbf{z}(t)=\phi(\mathbf{x}_{[s]})} = \mathbf{x}(t)$*.* ◇

Note that if $p = j = 0$, in Definition 8, the transformation is a bicausal change of coordinates, and the associated differential representation is characterized by a unimodular matrix.

2.4 Geometrical Settings

In this section, the mathematical framework of tools developed as an extension of the geometrical control theory is presented. In particular, the extended Lie bracket introduced in Califano *et al.* (2011a) has been the cornerstone for the solution of problems concerning change of coordinates to useful canonical forms for the control theory. The problem of equivalence of 1, under a bicausal change of coordinates, to a linear strongly controllable system is solved in Califano *et al.* (2010), while the case for equivalence with a weakly controllable system is attacked in Califano *et al.* (2011a). Solvability conditions for the case of nonlinear feedback linearization are provided in Califano and Moog (2011). Furthermore, the concept of accessibility is used in Califano and Moog (2012b) to find a delay-free representation of the system. It is known that single-input driftless systems without delays are not accessible when its dimension is higher than one. In Califano *et al.* (2013a) it is shown that for time-delay systems this is not the case, and a characterization of accessibility is made using the extended geometrical tools. Results on the equivalence via a bicausal change of coordinates to an observable weakly and strongly observable linear system up to input-output injection are presented in Califano *et al.* (2011b). The same problem is solved in Califano and Moog (2014) with the additional consideration of output transformation.

Consider the definition of the extended Lie bracket for time-delay systems.

Definition 10 *Let* $\mathbf{r}_1(\mathbf{x},\mathbf{u},\delta) = \sum_{j=0}^{s} \mathbf{r}_1^j(\mathbf{x},\mathbf{u})\delta^j$*, and* $\mathbf{r}_2(\mathbf{x},\mathbf{u},\delta) = \sum_{j=0}^{s} \mathbf{r}_2^j(\mathbf{x},\mathbf{u})\delta^j$*. The*

Table 1: Results on the change of coordinates reported in the literature using geometrical tools.

System is equivalent to the form	Characteristics of the system
$\dot{\mathbf{z}}(t) = \sum_{i=0}^{s} A_i\mathbf{z}(t-i) + \sum_{i=0}^{s} B_i u(t-i)$	Strongly controllable[a]. Weakly controllable[b]. By feedback linearization[c].
$\dot{\mathbf{z}}(t) = \sum_{i=0}^{s} A_i\mathbf{z}(t-i) + \varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$, $\mathbf{y}(t) = \sum_{i=0}^{s} C_i\mathbf{z}(t-i)$	Weakly observable[d].
$\dot{\mathbf{z}}(t) = \sum_{i=0}^{s} A_i\mathbf{z}(t-i) + \varphi(\bar{\mathbf{y}}_{[s]}, \mathbf{u}_{[s]})$, $\bar{\mathbf{y}}(t) = \sum_{i=0}^{s} C_i\mathbf{z}(t-i) = \bar{\varphi}(\mathbf{y}_{[s]})$	Weakly / strongly observable[e].
$\dot{\mathbf{z}}_{\mathbf{1}}(t) = \sum_{i=0}^{s} A_i\mathbf{z}_{\mathbf{1}}(t-i) + \sum_{i=0}^{s} B_i u(t-i)$ $\dot{\mathbf{z}}_{\mathbf{2}}(t) = \eta(\mathbf{x}_{\mathbf{1},[\mathbf{s}]}, \mathbf{x}_{\mathbf{2},[\mathbf{s}]})$, $\mathbf{y}(t) = \sum_{i=0}^{s} C_i\mathbf{z}_{\mathbf{1}}(t-i)$	by means of a feedback input $u(t) = \alpha(\mathbf{x}_{[\mathbf{s}]}) + \beta(\mathbf{x}_{[\mathbf{s}]})v(t)$[f]
[a]Califano *et al.* (2010), [b]Califano *et al.* (2011a), [c]Califano and Moog (2011), [d]Califano *et al.* (2011b), [e]Califano and Moog (2014), [f]Califano and Moog (2012a)	

extended Lie bracket $[\mathbf{r}_1^k(\cdot,\mathbf{u}), \mathbf{r}_2^k(\cdot,\mathbf{u})]_{E_i} \in \mathbb{R}^{(i+1)n}$, $i \geq 0$ *is defined as:*

$$\left[\mathbf{r}_1^k(\cdot,\mathbf{u}), \mathbf{r}_2^l(\cdot,\mathbf{u})\right]_{E_i} = \sum_{j=0}^{min(k,l,i)} (\left[\mathbf{r}_1^{k-j}(\cdot,\mathbf{u}), \mathbf{r}_2^{l-j}(\cdot,\mathbf{u})\right]_{E_0})^T \mid_{(x(-j),\mathbf{u}(-j))} \frac{\partial}{\partial x(t-j)} \tag{16}$$

with:

$$\left[\mathbf{r}_1^k(\cdot,\mathbf{u}), \mathbf{r}_2^l(\cdot,\mathbf{u})\right]_{E_0} = \sum_{i=0}^{k} \frac{\partial \mathbf{r}_2^l(\mathbf{x},\mathbf{u})}{\partial x(t-i)} \mathbf{r}_1^{k-i}(\mathbf{x}(-i),\mathbf{u}(-i)) - \sum_{i=0}^{l} \frac{\partial \mathbf{r}_1^k(\mathbf{x},\mathbf{u})}{\partial x(t-i)} \mathbf{r}_2^{k-i}(\mathbf{x}(-i),\mathbf{u}(-i)) \tag{17}$$

where $l, k, i \in \{0, 1, \ldots\}$. ◇

Some properties of the extended Lie bracket are:

P.i. Skew symmetry. $\left[\mathbf{r}_1^k(\cdot,\mathbf{u}), \mathbf{r}_2^l(\cdot,\mathbf{u})\right]_{E_i} = -\left[\mathbf{r}_2^l(\cdot,\mathbf{u}), \mathbf{r}_1^k(\cdot,\mathbf{u})\right]_{E_i}$.

P.ii. Let $k \leq l, k \leq \gamma, \quad \gamma \leq 0$

$$\left[\mathbf{r}_1^k, \mathbf{r}_2^l\right]_{E_\gamma} = \left[\mathbf{r}_1^k, \mathbf{r}_2^l\right]_{E_k} = \sum_{j=0}^{k} (\left[\mathbf{r}_1^{k-j}(\cdot,\mathbf{u}), \mathbf{r}_2^{l-j}(\cdot,\mathbf{u})\right]_{E_0})^T \mid_{(x(-j),\mathbf{u}(-j))} \frac{\partial}{\partial x(t-j)} \tag{18}$$

P.iii. Let $\hat{\gamma} \geq k + ps \geq 0,\ k \leq l$

$$\left[\mathbf{r}_1^{k+ps}, \mathbf{r}_2^{l+ps}\right]_{E_{\hat{\gamma}}} = \left[\mathbf{r}_1^{k+ps}, \mathbf{r}_2^{l+ps}\right]_{E_{k+ps}} = \sum_{j=0}^{k} (\left[\mathbf{r}_1^{k-j}(\cdot, \mathbf{u}), \mathbf{r}_2^{l-j}(\cdot, \mathbf{u})\right]_{E_0})^T \Big|_{(x(-\vartheta))} \frac{\partial}{\partial x(t-\vartheta)}, \quad (19)$$

where $\vartheta = j + ps$, $k \geq s$

P.iv. Let $k \leq l, k \leq \gamma, \quad \gamma \leq 0$

$$\left[\mathbf{r}_1^k, \mathbf{r}_2^l\right]_{E_\gamma} = \left[\mathbf{r}_1^k, \mathbf{r}_2^l\right]_{E_k} = 0, \qquad l - k > 2s \quad (20)$$

The set of vectors $\mathbf{r}_j(\mathbf{x}, \delta) = \sum_{i=0}^{s} \mathbf{r}_i^j(\mathbf{x})\delta^j$, $i = 1, \dots p$, are associated now with the infinite dimensional array $\Gamma = addcol\,(\Gamma_1, \dots, \Gamma_p)$ where

$$\Gamma_i = \left\{ \begin{array}{cccccccc} \mathbf{r}_i^1(\mathbf{x}) & \cdots & \mathbf{r}_i^s(\mathbf{x}) & 0 & \cdots & 0 & \cdots & \cdots \\ 0 & \mathbf{r}_i^1(\mathbf{x}(-1)) & \cdots & \mathbf{r}_i^s(\mathbf{x}(-1)) & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots & \\ 0 & \cdots & 0 & \mathbf{r}_i^1(\mathbf{x}(-q)) & \cdots & \mathbf{r}_i^s(\mathbf{x}(-p)) & 0 & \cdots \\ \vdots & \cdots & \vdots & \ddots & \ddots & \cdots & \ddots & \ddots \end{array} \right\}, \quad (21)$$

with $i = 1, \dots, p$. The columns of Γ form an infinite dimensional distribution

$$\Delta^\infty(\mathbf{x}_{[s]}) = \left\{ \begin{pmatrix} \mathbf{r}_1^1(\mathbf{x}) \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} \mathbf{r}_1^2(\mathbf{x}) \\ \mathbf{r}_1^1(\mathbf{x}(-1)) \\ \vdots \end{pmatrix}, \dots, \begin{pmatrix} \mathbf{r}_p^1(\mathbf{x}) \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} \mathbf{r}_p^2(\mathbf{x}) \\ \mathbf{r}_p^1(\mathbf{x}(-1)) \\ \vdots \end{pmatrix}, \dots \right\} \quad (22)$$

The elements of this space can be finitely represented, generating the rest of the elements by time, and row shifts, through the finite number of the infinite dimensional vectors $\bar{\mathbf{r}}_l^j(\mathbf{x}) = \sum_{i=0}^{j} \mathbf{r}_l^{j-i}(\mathbf{x}(-i)) \frac{\partial}{\partial \mathbf{x}(t-i)}$.

Consider the finite dimensional distributions

$$\begin{aligned}\Delta_i(\mathbf{x}_{[s]}) &= span_{\mathcal{K}}\left\{\hat{\mathbf{r}}_k^{\gamma}(\mathbf{x}_{[s]}) = \sum_{l=0}^{\gamma}\left(\mathbf{r}_k^{\gamma-l}(\mathbf{x}(-l))\right)^T \frac{\partial}{\partial \mathbf{x}(t-l)}, k \in [1,j], \gamma \in [0,i]\right\}, \\ \Delta_i'(\mathbf{x}_{[s]}) &= span_{\mathcal{K}}\left\{\hat{\mathbf{r}}_k'^{\gamma}(\mathbf{x}_{[s]}) = \sum_{l=0}^{min(\gamma,l)}\left(\mathbf{r}_k^{\gamma-l}(\mathbf{x}(-l))\right)^T \frac{\partial}{\partial \mathbf{x}(t-l)}, k \in [1,j], \gamma \in [0,i+s]\right\},\end{aligned} \tag{23}$$

with elements defined on $\mathbb{R}^{(i+1)n}$, for $i \in \mathbb{Z}^+$. By construction $\Delta_i \subseteq \Delta_i'$. Let $\rho_i = rank(\Delta'(\mathbf{x}_{[s]})$. Therefore, $\Delta'(\mathbf{x}_{[s]}) = span\{\upsilon_1, \ldots, \upsilon_{\rho_i}\} \subset \mathbb{R}^{(i+1)n}$ is not singular around $\mathbf{x}_0$. Note that Δ_i, and Δ_i' are endowed by a finite number of dimensional truncated column elements of Δ^{∞}. The extended Lie bracket is defined over the elements on Δ_i, $\hat{\mathbf{r}}_1^{\gamma}(\mathbf{x}_{[s]})$ and,$\hat{\mathbf{r}}_2^{\beta}(\mathbf{x}_{[s]})$ as

$$\left[\hat{\mathbf{r}}_1^{\gamma}(\mathbf{x}_{[s]}), \hat{\mathbf{r}}_2^{\beta}(\mathbf{x}_{[s]})\right] = \frac{\partial \hat{\mathbf{r}}_2^{\beta}(\mathbf{x}_{[s]})}{\partial \mathbf{x}_e}\hat{\mathbf{r}}_1^{\gamma}(\mathbf{x}_{[s]}) - \frac{\partial \hat{\mathbf{r}}_1^{\gamma}(\mathbf{x}_{[s]})}{\partial \mathbf{x}_e}\hat{\mathbf{r}}_2^{\beta}(\mathbf{x}_{[s]}). \tag{24}$$

Note that this definition is the well-known Lie bracket for delay-free vector spaces including the delayed variables. This implies that a Lie algebra can be constructed using the Lie bracket on the extended space defined above. The same can be said for Δ_i'. However, it is important to point out that the extended Lie bracket (16) defined over vectors in $\mathcal{K}^n(\delta]$ does not characterize a Lie algebra since the target set is $\mathbb{R}^{(i+1)n}$. Furthermore, the Jacobi identity is not fulfilled.

Chapter 3. Taylor–Series Approximation for a Class of Retarded Dynamical Systems

The objective of this chapter is twofold. First, recall stability conditions for a class of linear time-delay systems, and, second, to deal with an approximation based on a Taylor expansion of linear time-delay differential equations. The purpose of such strategy is aimed to the design of feedback control inputs. The use of approximations for this purpose is a widely used strategy for selection of a feedback control law for this kind of problem (Paraskevopoulos and Samiotakis (1994); Leipholz and Abdel-Rohman (1986); Seborg *et al.* (2010)). Nevertheless, instability of the approximated system notwithstanding the stability of the original time-delay system are reported in Mazanov and Tognetti (1974), Driver (1977), and Insperger (2015) for the Taylor series approximation, and in Silva *et al.* (2001) for the Padé approximation. In the following sections, a strategy to deal with the decision for the selection of coefficients for a control feedback law is proposed using the Taylor series of the delayed and advanced variables on the time-shifted time-delay differential equation. In this approach, the stability region of the approximation of the time-shifted time-retarded system is chosen to be a subset of the stability region of the original time-delay equation. This is done by the selection of the time-shift. The results presented in this chapters are used in the following chapters to find a delay-free approximated dynamics to allow the use of no-hereditary system theory. However, the analysis of such techniques has an importance in itself. The following motivational example is presented.

Consider the linearized equation (taken from Stépán (1989)) of the single degree of freedom mechanical system subjected to a delayed following force defined as

$$ml^2\ddot{q}(t) + (s - Fl)q(t) = -Flq(t - \tau), \tag{25}$$

where q represents the general coordinate, which is the angle with respect to the vertical axes, s is the torsional stiffness at the pin, m represents the mass at the end of the rod, τ is the constant time-delay of the bar angle, and F corresponds to the constant magnitude of the control input force. Note that, the feedback control law depends on the values at the past of the angle of the bar. This delay may be caused by a delayed measurement, transportation delay, or computational delay of the angle q. To find a value for the param-

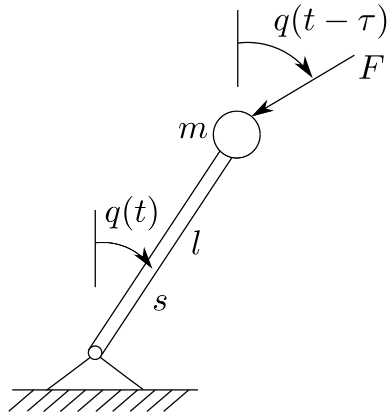


Figure 3: Mechanical system.

eter F which defines a stable solution for the equation (25) it is possible to use the next result taken from Cahlon and Schmidt (2004). Consider the time-delay system given by the equation

$$\ddot{y}(t) = p_1\dot{y}(t) + p_2\dot{y}(t-\tau) + q_1 y(t) + q_2 y(t-\tau). \tag{26}$$

It is assumed that $p_1 p_2 \geq 0$ and $q_1 q_2 < 0$, and set $A = \tau p_1$, $B = \tau^2 q_1$, $C = \tau p_2$, and $D = \tau^2 q_2$. Consider the next result from Cahlon and Schmidt (2004).

Theorem 1 *Assume $A = C = 0$, and $D > 0$ in equation* (26)*. Then the zero solution of* (26) *is asymptotically stable if and only if $B < 0$, and there exists $k \in \mathbb{Z}^+$ such that*

- $2k\pi < \sqrt{-B} < (2k+1)\pi$*, and*
- $D < min\,(-(2k)^2\pi^2 - B, (2k+1)^2\pi^2 + B)$*.*

This means that Theorem 1 establishes a stability region for the asymptotic stability of the system given by the equations

$$\ddot{y}(t) - q_1 y(t) - q_2 y(t-\tau) = 0 \tag{27}$$

with $q_2 > 0$. A particular useful stability region for this work is the one defined by $0 < \sqrt{-B} < \pi$ and $D < min\,(-B, \pi^2 + B)$, *i.e.* $k = 0$. In Figure 4, this region is presented for $\tau = 1$ units of time, and the index $k = 0$.

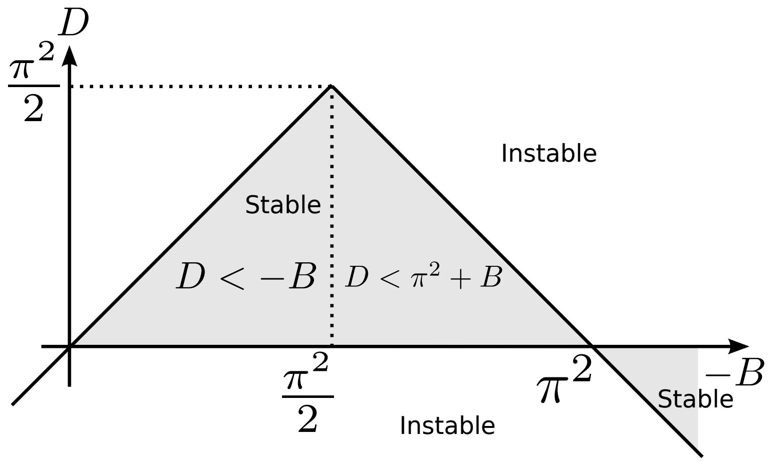


Figure 4: Stability region for a class of second-order linear time-delay system.

From Theorem 1 stability regions for the mechanical system (25) are defined by the next corollary

Corollary 2 *The zero solution of* (25) *is asymptotically stable if and only if* $-\frac{(s-Fl)\tau^2}{ml^2} < 0$*, and there exists* $k \in \mathbb{Z}^+$ *such that*

- $2k\pi < \sqrt{\frac{(s-Fl)\tau^2}{ml^2}} < (2k+1)\pi$*, and*
- $-\frac{F}{ml} < min\left(-(2k)^2\pi^2 + \frac{(s-Fl)\tau^2}{ml^2}, (2k+1)^2\pi^2 - \frac{(s-Fl)\tau^2}{ml^2}\right)$.

Corollary 2 establishes a stability region for the parameter F for equation (25). Nevertheless, despite that the stability regions are defined, the behavior of the system, once the selection of F fulfills the stability conditions, can not be defined only by this result. The study of approximations to linear delay-free systems is a possible solution for this problem.

3.1 Taylor Expansion of the Linear first-order Pure-Delay System.

In this section the dynamical system given by the equation

$$\dot{x}(t) = -ax(t-\tau) \tag{28}$$

is considered, with a a real number. The following theorem is taken from Matsunaga (2007).

Theorem 3 *The zero solution of* (28) *is asymptotically stable if and only if*

$$0 < a < \frac{\pi}{2\tau}. \tag{29}$$

A strategy that can be followed to approximate equation (28) is by means of the Taylor series expansion of the delayed variable $x(t-\tau)$ defined as

$$x(t-\tau) \simeq x(t) - \tau\dot{x}(t) + \frac{\tau^2}{2!}\ddot{x}(t) - \frac{\tau^3}{3!}x^{(3)}(t) + \dots \tag{30}$$

The accuracy of this approach is questioned by the fact that, if more than three terms of the Taylor series (for this particular equation) are taken, the resulting equation is unstable. This inconsistency implies that for a truncation of order higher than two the system is no longer an approximation of the original system. Furthermore, as it is shown in Table 2, the stability regions for lower truncations do not match with the one given by Theorem 3.

Table 2: Stability regions of several truncations of the delayed variable Taylor approximation.

Equation	Stability regions
$\dot{x}(t) = -a\left(x(t)\right)$	$a > 0$
$\dot{x}(t) = -a\left(x(t) - \tau\dot{x}(t)\right)$	$0 < a < \frac{1}{\tau}$
$\dot{x}(t) = -a\left(x(t) - \tau\dot{x}(t) + \frac{\tau^2}{2!}\ddot{x}(t)\right)$	$0 < a < \frac{1}{\tau} = \frac{2}{2\tau}$
$\dot{x}(t) = -ax(t-\tau)$	$0 < a < \frac{\pi}{2\tau}$
$\dot{x}(t) = -a\left(x(t) - \tau\dot{x}(t) + \frac{\tau^2}{2!}\ddot{x}(t) - \frac{\tau^3}{3!}x^{(3)}(t)\right)$ and truncations of higher order	Unstable

The inconsistency in the stability regions mentioned above raises the discussion about the possibility of use of the Taylor series expansion to find a free of delay approximation for linear hereditary systems. In the following sections, it will be introduced a strategy, based on Taylor series approximations combined with the use of the advance operator, that shows a wider range of consistency in the stability regions.

3.1.1 Taylor Approximation of the Advanced Pure-Delay First-Order Linear System Shifted by the Amount of the Delay.

In this section the Taylor series expansion of the advance variable $x(t+\tau)$ is proposed instead of the expansion of the delayed variable. This is done by the use of a time advance shift on the equation 28. Let us rewrite equation (28) as

$$\dot{x}(t+\tau) + ax(t) = 0. \tag{31}$$

Consider the expansion given by

$$x(t+\tau) = x(t) + \tau\dot{x}(t) + \frac{\tau^2}{2}\ddot{x}(t) + \frac{\tau^3}{6}x^{(3)}(t) + \ldots. \tag{32}$$

If (32) is substituted into the equation (28), the Taylor series expansion truncation by the first, second, and third order differential equation and using the Routh-Hurwitz criterion the resultant stability conditions are presented in Table 3.

Table 3: Stability regions for the fordwarded τ shift variable Taylor series approximation.

Equation	Stability conditions
$\tau\ddot{x}(t) + \dot{x}(t) + ax(t) = 0$	$a > 0$
$\frac{\tau^2}{2}x^{(3)} + \tau\ddot{x}(t) + \dot{x}(t) + ax(t) = 0$	$0 < a < \frac{2}{\tau}$
$\frac{\tau^3}{6}x^{(4)} + \frac{\tau^2}{2}x^{(3)} + \tau\ddot{x}(t) + \dot{x}(t) + ax(t) = 0$	$0 < a < \frac{4}{3\tau}$

From this results the following result is stated

Proposition 4 *If the system described by the equation*

$$\frac{\tau^3}{6}x^{(4)} + \frac{\tau^2}{2}x^{(3)} + \tau\ddot{x}(t) + \dot{x}(t) + ax(t) = 0 \tag{33}$$

is asymptotically stable, then system (28) *is asymptotically stable.*

Note that, for a fifth order expansion the resultant system is unstable as can be seen in system

$$\frac{\tau^6}{720}x^{(7)}(t)+\frac{\tau^5}{120}x^{(6)}(t)+\frac{\tau^4}{24}x^{(5)}(t)+\frac{\tau^3}{6}x^{(4)}(t)+\frac{\tau^2}{2}x^{(3)}(t)+\tau\ddot{x}(t)+\dot{x}(t)+ax(t) \tag{34}$$

that has the Routh-Hurwitz array

$$\begin{bmatrix} \frac{\tau^6}{720} & \vdots & \frac{\tau^4}{24} & \frac{\tau^2}{2} & 1 \\ \frac{\tau^5}{120} & \vdots & \frac{\tau^3}{6} & \tau & a \\ \frac{\tau^4}{72} & \vdots & \frac{\tau^2}{3} & -\frac{a\tau-6}{6} & 0 \\ -\frac{\tau^3}{30} & \vdots & \frac{a\tau^2+4\tau}{10} & a & 0 \\ \frac{a\tau^3+12\tau^2}{24} & \vdots & \frac{a\tau+4}{4} & 0 & 0 \\ \frac{a^2\tau^3+18a\tau^2+56\tau}{10a\tau+120} & \vdots & a & 0 & 0 \end{bmatrix}.$$

Since $-\frac{\tau^3}{30} < 0$ is always true, this array has a change of sign on the fourth row so the system (34) is unstable despite the stability of (28).

3.1.2 Taylor Approximation by Forward Shifting a Half of the Delay.

The stability analysis of the approximation of the equation (35) is valid only on the interval $0 < a < \frac{8/3}{2\tau}$. In contrast, the stability interval presented by equation (28) has an stability interval defined on $0 < a < \frac{\pi}{2\tau}$. Now equation (28) is time-shift advanced resulting in the equation

$$\dot{x}(t+\frac{\tau}{2})+ax(t-\frac{\tau}{2})=0. \tag{35}$$

Substituting the advanced and delayed variables by their corresponding Taylor series, up to order 2, the new expression is written as

$$\frac{\tau^2}{8}x^{(3)}(t)+\left(\frac{\tau}{2}+a\frac{\tau^2}{8}\right)\ddot{x}(t)+\left(1-a\frac{\tau}{2}\right)\dot{x}(t)+ax(t). \tag{36}$$

The Routh-Hurwitz array is described by

$$\begin{bmatrix} \frac{\tau^2}{8} & \vdots & 1-\frac{a\tau}{2} \\ \frac{a\tau^2}{8}+\frac{\tau}{2} & \vdots & a \\ -\frac{a^2\tau^2+4a\tau-8}{2a\tau+8} & \vdots & 0 \\ a & \vdots & 0 \end{bmatrix}.$$

From the Routh-Hurwithz criteria, the stability conditions for (36) are given by the inequalities

$$\begin{aligned} a &> 0 \\ \tau^2 a^2 + 4\tau a - 8 &< 0, \end{aligned}$$

and then

$$0 < a < \frac{-4+4\sqrt{3}}{2\tau}. \tag{37}$$

Which is a subset of the stability region of the original time-delay system. This conditions allows to state the following result.

Proposition 5 *If the system described by the equation* (36) *is asymptotically stable, then system* (28) *is asymptotically stable.*

Proof. The sufficiency is given since (37) defines a subset of the stability conditions of the pure-delay first-order system given in Theorem 3. ■

3.1.3 Taylor Approximation by Forward Shifting a Customized Fraction of the Delay.

As it can be seen in the previous sections, the accuracy of the stability region of the Taylor expansion approximation is improved, with respect to the original time-delay equation (28), by shifting forward a fraction of the delay. In this section, it is proved that the stability regions of the original, and approximate equations can be paired enough by a proper choice of the delay fraction to be advanced.

Let us time-shift equation (28) by $\frac{\tau}{\kappa}$. Equation (28) is rewritten then as:

$$\dot{x}(t+\frac{\tau}{\kappa}) + ax(t-\tau+\frac{\tau}{\kappa}) = 0, \tag{38}$$

where $\tau \geq \frac{\tau}{\kappa}$.

At this point it is important to show that, even if the stability parametric regions match exactly, the truncation order affects the accuracy of the approximation. Consider the Taylor series expansion truncated at the second element of equation (38) described as

$$\frac{\tau}{\kappa}\ddot{x}(t) + \left(1 - a\tau\frac{\kappa-1}{\kappa}\right)\dot{x}(t) + ax(t) = 0 \tag{39}$$

Consider now the following Lemma

Lemma 6 *Consider system* (39)*, and let* $a_0 \in \mathbb{R}$*,* $\tau \in \mathbb{R}^+$*. Then, there exists only one* $k = \bar{\kappa} \in \mathbb{R}$ *for which necessary and sufficient conditions for the asymptotic stability of* (39) *are defined by the inequality*

$$0 < a < \frac{\pi}{2\tau}. \tag{40}$$

Proof. By Routh-Hurwitz criteria, necessary and sufficient conditions for the asymptotic stability of (39) are given

$$\begin{aligned} \tfrac{\tau}{\kappa} &> 0, \\ \left(1 - a\tau\tfrac{\kappa-1}{\kappa}\right) &> 0, \\ a_0 &> 0, \end{aligned}$$

That implies $\kappa > 0$, and

$$-a\tau(\kappa-1) > -\kappa.$$

There exist three possibilities $\kappa < 1$, $\kappa = 1$, and $\kappa > 1$. For $\kappa < 1$, $1-\kappa > 0$, and then

$$a > -\frac{\kappa}{1-\kappa}\frac{1}{\tau},$$

where $\frac{\kappa}{1-\kappa}\frac{1}{\tau}$ is positive. Such a choice implies stability conditions $a_0 > 0 > -\frac{\kappa}{1-\kappa}\frac{1}{\tau}$ that are not the ones proposed by Lemma 6. $\kappa = 1$ implies the same stability condition $a_0 > 0$ and correspond to the case of equation (35). For $\kappa > 1$ the inequality

$$a < \frac{\kappa}{\kappa-1}\frac{1}{\tau}$$

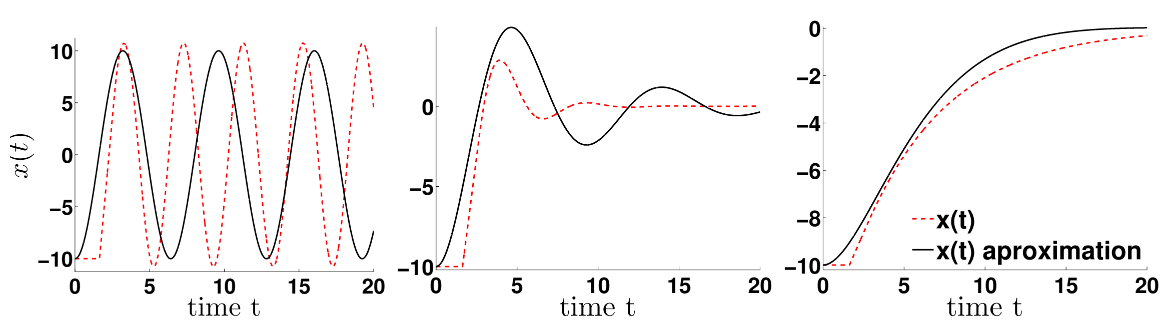
stands. This means that, conditions for the stability of (38) can be stated as

$$0 < \tau < \frac{\kappa}{\kappa-1}\frac{1}{\tau}.$$

Since $\frac{\kappa}{\kappa-1}\frac{1}{\tau} = \frac{\pi}{2}$ has only one solution, assigning $\bar{\kappa} = \kappa = \frac{\pi}{\pi+2}$, the proof is completed. ■

Lemma 6 states that the stability regions of equations (28), and (39) are equal. Notwithstanding, in Figure 5, where numerical solution of both equations are displayed, it is shown a notable difference between the original equation and the approximated one even if the stability regions match. Numerical solutions presented on Figure 5 consider values of $\tau = 1$, $\kappa = \frac{\pi}{\pi+2}$ and several values of the parameter a. Note that, to establish a relationship with the solution of (38), the solution of the differential equation (28) is time-shifted $\frac{\pi+2}{\pi}$ units of time.

As it is shown later on, if a higher number of elements of the truncated expansion are



(a) Solution for $a = \frac{\pi}{2}$ (b) Solution for $a = \frac{\pi}{4}$ (c) Solution for $a = \frac{\pi}{20}$

Figure 5: Truncation at second element of the Taylor series for a customized forward shift.

admitted, the approximated signal presents an smaller error with respect to the signal from the time-delay system.

Consider a truncation at the third element of the Taylor series expansion on system (38) that lead to the delay-free differential equation

$$\frac{\tau^2}{2\kappa^2}x^{(3)}(t)+\left(\frac{a\left(\frac{\tau}{\kappa}-\tau\right)^2}{2}+\frac{\tau}{\kappa}\right)\ddot{x}(t)+\left(a\left(\frac{\tau}{\kappa}-\tau\right)+1\right)\dot{x}(t)+ax(t)=0. \tag{41}$$

The following proposition is formulated about the asymptotic stability of system (41).

Proposition 7 *Consider system* (41), *let* $a_0\in\mathbb{R}$, $\tau\in\mathbb{R}^+$. *Then, there exists* $k=\bar{\kappa}\in\mathbb{R}$ *for which necessary and sufficient conditions for the asymptotic stability of* (41) *are defined by the inequality*

$$0<a<\frac{\pi}{2\tau}. \tag{42}$$

Proof. The proof comes from the Routh-Hurwitz criteria on the equation on (41). The Routh-Hurwithz array of equation (41) is

$$\begin{bmatrix} \frac{\tau^2}{2\kappa^2} & \vdots & a\left(\frac{\tau}{\kappa}-\tau\right)+1 \\ \frac{a\left(\frac{\tau}{\kappa}-\tau\right)^2}{2}+\frac{\tau}{\kappa} & \vdots & a \\ -\frac{\left(a^2\kappa^3-3a^2\kappa^2+3a^2\kappa-a^2\right)\tau^2+\left(-a\kappa^3+4a\kappa^2-2a\kappa\right)\tau-2\kappa^2}{(a\kappa^3-2a\kappa^2+a\kappa)\tau+2\kappa^2} & \vdots & 0 \\ a & \vdots & 0 \end{bmatrix}$$

Since, from the second row of the Routh-Hurwitz array, $\left(a\kappa^3-2a\kappa^2+a\kappa\right)\tau+2\kappa^2>0$, from the third row of this array we have that

$$\left(\kappa^3-3\kappa^2+3\kappa-1\right)\tau^2a^2+\left(-\kappa^3+4\kappa^2-2\kappa\right)\tau a-2\kappa^2<0. \tag{43}$$

From the rest of the Routh-Hurwitz conditions, the inequality

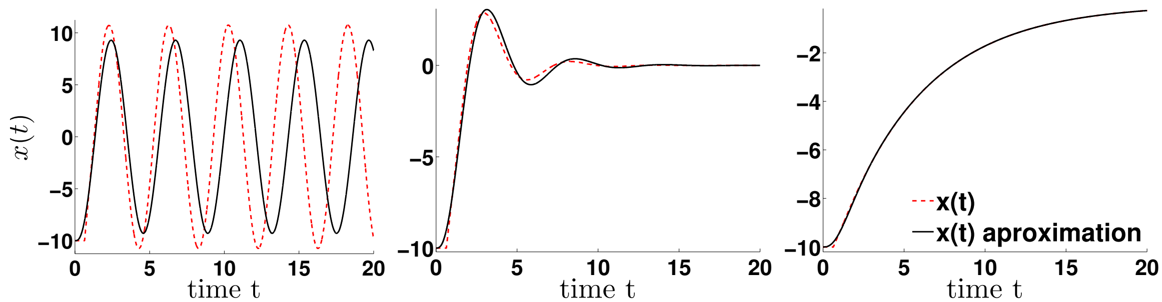
$$0<a<\frac{\kappa\sqrt{\left(\kappa^2-2\kappa+2\right)\left(\kappa^2+2\kappa-2\right)}+\kappa^3-4\kappa^2+2\kappa}{\left(\kappa^3-3\kappa^2+3\kappa-1\right)}\frac{1}{2\tau}=\frac{\pi}{2\tau}, \tag{44}$$

must be fulfilled. Since

$$\frac{\kappa\sqrt{(\kappa^2-2\kappa+2)(\kappa^2+2\kappa-2)}+\kappa^3-4\kappa^2+2\kappa}{(\kappa^3-3\kappa^2+3\kappa-1)}=\pi$$

has at least one real solution, $\bar{\kappa} \simeq 1.5669$, the Proposition is proved. ■

Figure 6 shows the solution of equation (41) for $\tau = 1$, $\kappa = 1.5669$ and different values of the parameter a. The $\frac{1}{1.5669}$ time-shifted solution of equation (28) is included for the corresponding value of a in seek of comparison.



(a) Solution for $a = \frac{\pi}{2}$ (b) Solution for $a = \frac{\pi}{4}$ (c) Solution for $a = \frac{\pi}{20}$

Figure 6: Truncation at third element of the Taylor series for a customized forward shift.

The above allows to present the following result.

Proposition 8 *Let $\kappa = \bar{\kappa}$ satisfy Proposition 7. Then, the time-delay linear system described by the equation* (28) *is asymptotically stable if and only if system* (41) *is asymptotically stable.*

3.2 Taylor Approximation for a Class of Linear second-order Time-Delay Differential Equation.

Let us consider the $\frac{\tau}{\kappa}$ shift of the equation (27).

$$\ddot{y}(t+\frac{\tau}{\kappa}) - q_1 y(t+\frac{\tau}{\kappa}) - q_2 y(t-\tau+\frac{\tau}{\kappa}) = 0 \tag{45}$$

The approximated system, substituting the advanced and delayed variables by its Taylor series expansion up to order two, is given by the equation

$$\frac{\tau^2}{2\kappa^2}y^{(4)}(t)+\frac{\tau}{\kappa}y^{(3)}(t)+\alpha(\tau,\kappa,q_1,q_2)\ddot{y}(t)+\left(q_2\left(\tau-\frac{\tau}{\kappa}\right)-\frac{q_1\tau}{\kappa}\right)\dot{y}(t)-(q_1+q_2)y(t)=0, \quad (46)$$

where $\kappa \in \mathbb{R}$, and $\alpha(\tau,\kappa,q_1,q_2)=\left(1-\frac{q_2\left(\frac{\tau}{\kappa}-\tau\right)^2}{2}-\frac{q_1\tau^2}{2\kappa^2}\right)$.
Conditions for asymptotic stability can be summarized in the following Lemma.

Lemma 9 *Suppose that $q_2>0$, and $\kappa>1$. System* (46) *is asymptotically stable if and only if conditions*

i. $0<-q_1\tau^2<\frac{2\kappa^2}{\kappa-1}$,

ii. $q_2\tau^2<min\left(-q_1\tau^2, \frac{2\kappa^2}{(1-\kappa)^2}-\frac{q_1\tau^2}{(1-\kappa)}\right)$,

hold.

Proof. Sufficiency and necessity are derived as follows. Consider the following Routh-Hurwitz array

$$\begin{bmatrix} \frac{\tau^2}{2\kappa^2} & \vdots & -\frac{\left(\kappa^2-2\kappa+1\right)\tau^2q_2+q_1\tau^2-2\kappa^2}{2\kappa^2} & -q_2-q_1 \\ \frac{\tau}{\kappa} & \vdots & \frac{(\kappa-1)\tau q_2-q_1\tau}{\kappa} & 0 \\ -\frac{(\kappa-1)\tau^2q_2-2\kappa}{2\kappa} & \vdots & -q_2-q_1 & 0 \\ \frac{\left(\kappa^2-2\kappa+1\right)\tau^3{q_2}^2+\left((-\kappa+1)q_1\tau^3-2\kappa^2\tau\right)q_2}{(\kappa^2-\kappa)\tau^2q_2-2\kappa^2} & \vdots & 0 & 0 \\ -q_2-q_1 & \vdots & 0 & 0 \end{bmatrix}.$$

Stability conditions are summarized by the following inequalities

$$\begin{aligned} q_2 &> 0 & q_1+q_2 &< 0 \\ \frac{\tau}{\kappa} &> 0 & (\kappa-1)q_2\tau^2-2\kappa &< 0 \\ -\frac{q_2\tau^2(1-\kappa)^2}{2\kappa^2}+1-\frac{q_1\tau^2}{2\kappa^2} &> 0 & (\kappa^2-2\kappa+1)\tau^2q_2+(-\kappa+1)q_1\tau^2-2\kappa^2 &< 0 \\ \frac{q_2\tau(\kappa-1)}{\kappa}-\frac{q_1\tau}{\kappa} &> 0 \end{aligned}$$

That implies $\kappa > 0$, $q_1 < 0$, and

$$q_2\tau^2 < \frac{2\kappa^2 - (1-\kappa)q_1\tau^2}{(1-\kappa)^2} < \frac{2\kappa^2 - q_1\tau^2}{(1-\kappa)^2}, \tag{47}$$

$$q_2\tau^2 < -q_1\tau^2, \tag{48}$$

$$q_2\tau^2 < \frac{2\kappa}{(\kappa-1)}, \tag{49}$$

$$q_2\tau^2 > \frac{q_1\tau^2}{(\kappa-1)}. \tag{50}$$

From (47), (48), and (49),

$$q_2\tau^2 < min\left(\frac{2\kappa}{(\kappa-1)}, -q_1\tau^2, \frac{2\kappa^2 - (1-\kappa)q_1\tau^2}{(1-\kappa)^2}\right). \tag{51}$$

The inequality $-q_1\tau^2 < \frac{2\kappa^2-(1-\kappa)q_1\tau^2}{(1-\kappa)^2}$ holds for

$$-q_1\tau^2 < \frac{2\kappa}{(\kappa-1)}, \tag{52}$$

and the inequality $\frac{2\kappa^2-(1-\kappa)q_1\tau^2}{(1-\kappa)^2} < \frac{2\kappa}{(\kappa-1)}$ for

$$-q_1\tau^2 > \frac{2\kappa}{(\kappa-1)}, \tag{53}$$

so $min\left(-q_1\tau^2, \frac{2\kappa^2-(1-\kappa)q_1\tau^2}{(1-\kappa)^2}\right) < min\left(\frac{2\kappa}{(\kappa-1)}, -q_1\tau^2, \frac{2\kappa^2-(1-\kappa)q_1\tau^2}{(1-\kappa)^2}\right)$. That proves *i*.

Finally, since $q_2\tau^2 > 0$ (50) always hold true, and, from (47),

$$0 < -q_1\tau^2 < \frac{2\kappa^2}{\kappa-1}. \tag{54}$$

■

Conditions of the proof can be illustrated by Figure 7.

Note that a proper selection of κ in Lemma 9 can lead to a stability region of (46) which is a subset, in the parametric space, of the regions established in Theorem 1 for system (27). This is illustrated by the following corollaries.

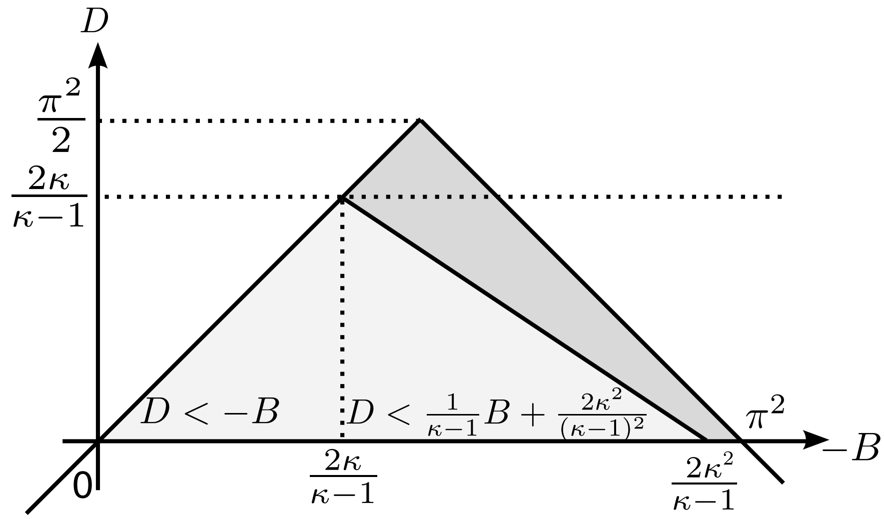


Figure 7: Stability region for a Taylor series approximation family of equations of a class of second-order time-delay equation.

Corollary 10 *Let* $\kappa = 2$*, in equation* (46)*. Then, system* (46) *is asymptotically stable if and only if*

i. $0 < -q_1\tau^2 < 8$*,*

ii. $q_2\tau^2 < min\,(-q_1\tau^2, 8 + q_1\tau^2)$*.*

The stability region for equation (46) with conditions stated in Corollary (10) are graphically illustrated in the Figure 8.

Corollary 10 leads to the following immediate result.

Proposition 11 *Let* $\kappa = 2$*, in equation* (46)*. If the system described by the equation* (46) *is asymptotically stable, then system* (27) *is asymptotically stable.*

In Figure 9, numerical results of the solutions, with a time-delay $\tau = 1$, considering different values of q_1 and q_2 are located on the parametric plane that describe the stability regions defined by Corollary 10 clear gray, and for 1 dark gray.

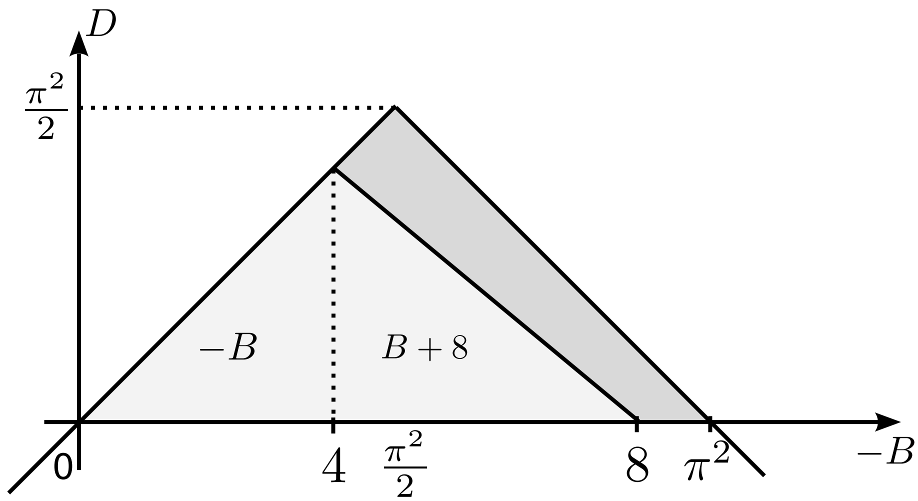


Figure 8: Parametric plane stability regions for an approximation of the $\tau/\bar{\kappa}$-forward time shift second-order equation.

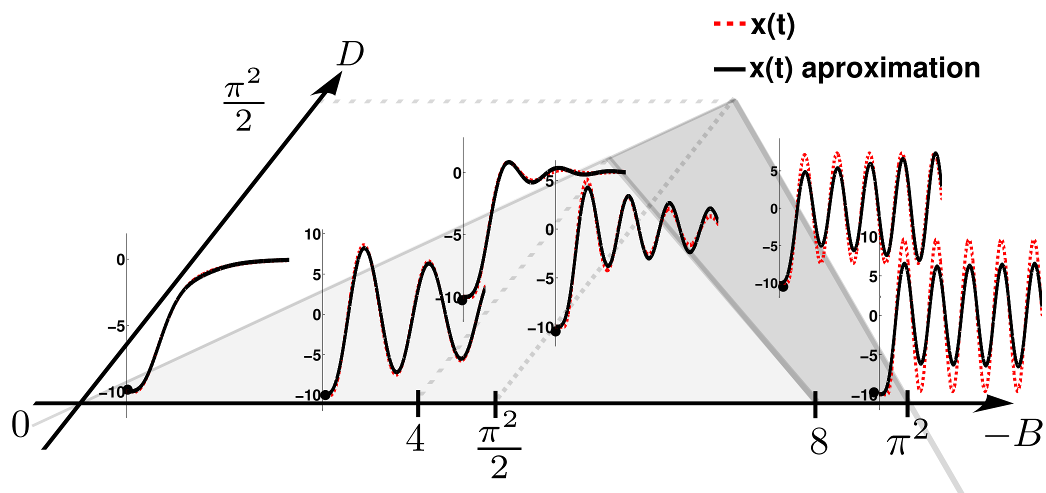


Figure 9: Numerical results on the parametric plane for an approximation of the $\tau/2$-forward time shift second-order equation.

Now consider a different selection for the parameter κ. Define

$$\frac{2\kappa^2}{\kappa-1}=\pi^2,$$

to match the base of the triangle that defines the stability region $0<\sqrt{-B}<\pi$ from Theorem 1, with the one from Lemma 9. The above implies two possible solutions

$$\bar{\kappa}_{1,2}=\frac{\pi^2\pm\pi\sqrt{\pi^2-8}}{4}.$$

Since solution $\bar{\kappa}_2=\frac{\pi^2-\pi\sqrt{\pi^2-8}}{4}$ defines a stability region of the delay-free system (46) with points outside the stability region of equation (27). Consider the following corollary for solution $\bar{\kappa}_1=\frac{\pi^2+\pi\sqrt{\pi^2-8}}{4}$.

Corollary 12 *Let* $\kappa=\frac{\pi^2+\pi\sqrt{\pi^2-8}}{4}$*, in equation* (46)*. Then, system* (46) *is asymptotically stable if and only if*

i. $0<-q_1\tau^2<\pi^2$*,*

ii. $q_2\tau^2<min\left(-q_1\tau^2,\frac{2\pi^3\sqrt{\pi^2-8}+2\pi^4-8\pi^2}{(\pi^3-4\pi)\sqrt{\pi^2-8}+\pi^4-8\pi^2+8}+\frac{4q_1\tau^2}{\pi\sqrt{\pi^2-8}+\pi^2-4}\right)\simeq min\left(-q_1\tau^2,3.9+0.4q_1\tau^2\right).$

In Figure 10 the stability region defined by Corollary 12 is presented and superpose on the one described by Theorem 1. This allows to illustrate that the region defined from Corollary 12 is a subset of the one defined by Theorem 1.

Proposition 13 *Let* $\kappa=\frac{\pi^2+\pi\sqrt{\pi^2-8}}{4}$*, in equation* (46)*. If the system described by the equation* (46) *is asymptotically stable, then system* (27) *is asymptotically stable.*

In Figure 11, numerical results of the solutions for time-delay $\tau=1$ are located on the parametric plane that describe the stability regions defined by Corollary 12 in blue, and for 1 in red. This is done for different values of q_1 and q_2.

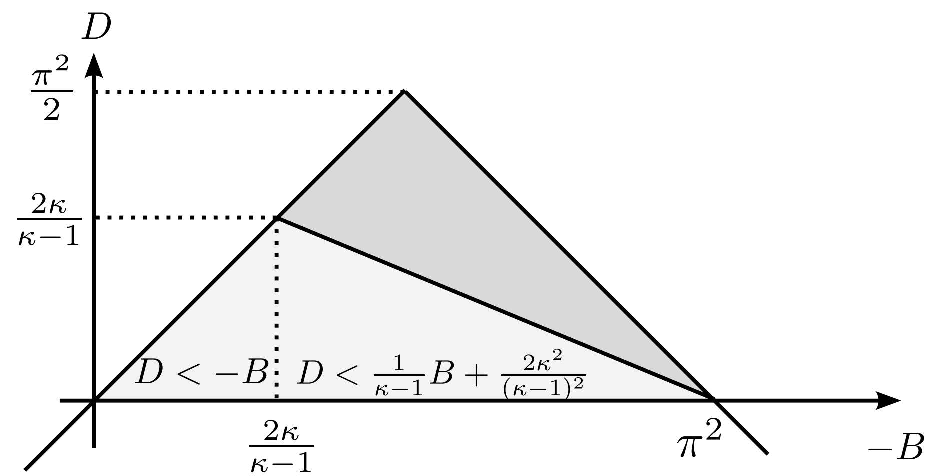


Figure 10: Parametric plane stability regions for an approximation of the $\tau/\bar{\kappa}$-forward time shift second-order equation.

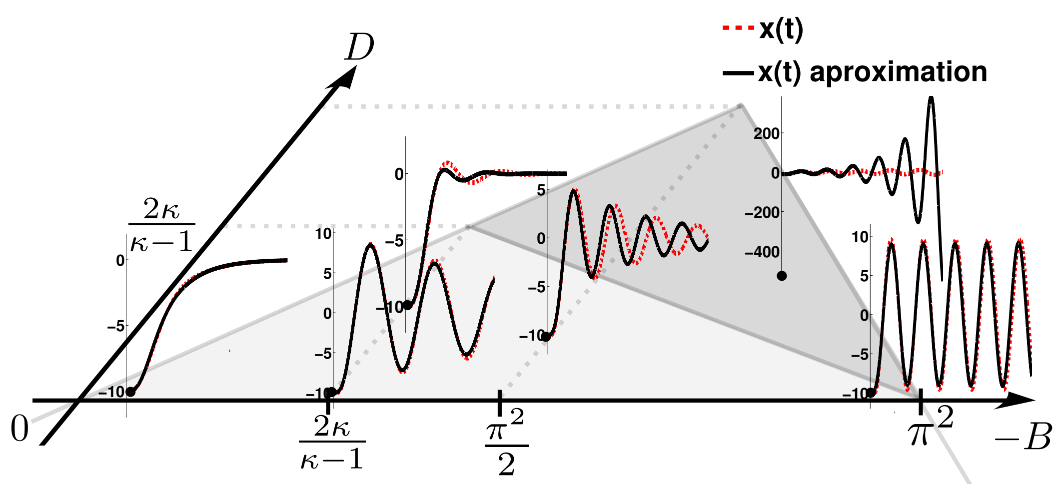


Figure 11: Numerical results on the parametric plane for an approximation of the $\tau/\bar{\kappa}$-forward time shift second-order equation.

3.3 Feedback Stabilization of a Class of Time-Delay Linear System Using the Taylor Expansion Approximation.

In this section, the approximation theory presented throughout this chapter is used in the selection of coefficients of a feedback control law for a second-order linear time-delay system.

Consider the SISO delay free linear system given by the equation

$$x^{(n)}(t) + \sum_{i=0}^{n-1} a_i x^{(i)}(t) = u(t), \tag{55}$$

where $u(t) \in \mathbb{R}$ is the input, $x(t) \in \mathbb{R}$ is the state variable, and a_i, $i = 0, \ldots, n-1$ are constant coefficients. However, if the state of the system is available only after some (constant) delay τ, then the actual state feedback which is plugged into the system is

$$u(t) = -\sum_{i=0}^{n-1} k_i x^{(i)}(t-\tau), \quad \forall \quad \tau \geq 0, \tag{56}$$

yielding the closed-loop dynamics

$$x^{(n)}(t) + \sum_{i=0}^{n-1} a_i x^{(i)}(t) + \sum_{i=0}^{n-1} k_i x^{(i)}(t-\tau) = 0, \tag{57}$$

which is a delayed-differential equation. The delay τ cannot be neglected in the stability analysis of (57). Nevertheless, the use of Taylor expansion (30) can lead to instability of the systems as can be presumed from Table 2. Consider the case where $k_0 \neq 0$, and $k_i = 0$ for $i = 1, \ldots, n-1$. If $m \gg n$, then the expansion of the delayed variable at the m-order truncation for equation (55) is given by

$$\begin{aligned} &\textstyle\sum_{i=n+1}^{m} \left(k_0 \frac{(-1)^i}{i!}\tau^i\right) x^{(i)}(t) + \left(1 + k_0 \frac{(-1)^n}{n!}\tau^n\right) x^{(n)}(t) + \\ &\textstyle\sum_{i=0}^{n-1} \left(a_i + k_0 \frac{(-1)^{(i)}}{i!}\tau^i\right) x^{(i)}(t) = 0. \end{aligned} \tag{58}$$

Now, the following result shows that (58) is unstable independently of the system parameters for a truncation greater than $n+2$.

Proposition 14 *For any $\tau > 0$, $k_0 \neq 0$, and $m \geq n+2$ system* (58) *is unstable.*

Proof. The first two coefficients of (58) are $k_0 \frac{(-1)^m}{m!} \tau^m$, and $k_0 \frac{(-1)^{m-1}}{(m-1)!} \tau^{m-1}$, which necessarily have opposite signs for $k_0 \neq 0$ and $\tau > 0$. So, there is at least one unstable root which makes (58) unstable. ■

Now theory developed in this section is proposed for the design of closed loop control inputs for two different control systems. Consider the first-order time-delay system

$$\begin{aligned} \dot{x}(t) &= \bar{a}x(t-\tau) + bu(t-\tau_2), \\ y(t) &= cx(t-\tau_3), \end{aligned} \tag{59}$$

with $\tau_2 + \tau_3 < \tau$. Then, consider the input

$$u(t) = \frac{k-\bar{a}}{cb} y(t-\tau_4) = \frac{k-\bar{a}}{b} x(t-\tau_3-\tau_4), \tag{60}$$

with $\tau_4 = \tau - \tau_2 - \tau_3$, that leads to a closed-loop equation

$$\dot{x}(t) = kx(t-\tau). \tag{61}$$

Then, results given in Subsection 3.1.3 can be applied for the selection of k.

Proposition 15 *Let system* (41) *be asymptotically stable at the origin $x = 0$, with κ defined as in Lemma 6, then the closed-loop system form of equation* (59) *is asymptotically stable at the origin $x = 0$ with a control law defined by* (60) *with $k = -a$.*

Proof. It comes directly from (61), and Proposition 8. ■

Now consider the second-order time-delay linear system given by

$$\begin{aligned} \ddot{x}(t) &= a_1 x(t) + a_2 x(t-\tau) + bu(t-\tau_2) \\ y(t) &= cx(t-\tau_3) \end{aligned} \tag{62}$$

with $\tau_2 + \tau_3 < \tau$. The feedback input

$$u(t) = \frac{k - a_2}{cb} y(t - \tau_4) = \frac{k - a_2}{b} x(t - \tau_3 - \tau_4), \tag{63}$$

with $\tau_4 = \tau - \tau_2 - \tau_3$, defines a closed loop time-delay differential equation

$$\ddot{x}(t) = a_1 x(t) + kx(t - \tau). \tag{64}$$

Now, the following proposition can be stated for the stability of system (62) in its closed loop form.

Proposition 16 *Let system* (46) *be asymptotically stable at the origin* $x = 0$ *with* κ *defined as in Propositions 11 or 13, and let it be an approximation of the closed loop equation* (63) *with* $a_1 = q_1$ *and* $k = q_2$*. Then, the closed loop system form of equation* (62) *is asymptotically stable at the origin* $x = 0$ *with a control law defined by* (63)*.*

Proof. It comes directly from (64), and Propositions 11 and 13. ■

3.4 Discussion.

In this chapter, general conditions on the truncation order of the Taylor expansions to ensure the stability of the approximating ordinary differential equations where presented. Furthermore, Taylor series approximations for a class of time-delay equations were successfully defined. These approximations are computed by forward-shifting the original equations, and substituting the delayed and advanced state variables by the first terms of their Taylor series expansion. The resultant equations obtained throughout this procedure are linear delay-free ordinary delay equations with an extended derivative order. It was proven that the stability regions, in the parametric space, of the approximated system are subsets of the stability regions of the original studied time-delay systems. In the case of single delay first-order differential equation it was possible to find a unique parameter value, for the forward-shift system, to match the stability regions for a first, and second-order truncation of the Taylor series expansion. Numerical results shown that the solution for the second-order truncation fits better with the aftereffect system than the one of the expansion truncated at the second element. It is important to remark that both approximations were the result of a parameter selection with different values. Since the second-order truncation presented a satisfactory response, in this work higher order truncations where not considered. Because the difference in the response of both approximations are considerable an error criteria is not considered in this work. Similar inferences can be done for the family of approximations of the studied second-order delay differential equation. Further observations about the stability region described by this delay-free equations should be done. While the characterized regions of the approximated delay-free equations do not match exactly with the produced by the second-order delay differential equation, the technique used allows to choose, excluding a certain region, the most convenient approximated equation. The manner to choice of such equation is beyond the scope of this work, however, the use of optimization techniques may lead to a solution. Finally, in Section 3.3 the results exposed in the previous sections are used in the control design for a class of control systems. The results of this chapter can be used in the strategy described in Figure 2 from page 6.

Chapter 4. Geometrical Theory

In the present chapter, it is presented the enumeration of several contributions to the geometrical theory introduced by Califano *et al.* (2011a) used in several linearization results, as shown in Table 1. Particularly, a connection between the extended system with the step method of solution for functional differential equations is introduced. A remarkable consequence of this relationship is that it makes clear that the initial conditions of the extended system are the same as the ones of the original dynamics plus the solution for $t > 0$ of this equation. Also, through a change of variable in the truncated system, sufficient conditions for the computation of the relative degree of a time-delay system are stated. This is possible considering the variables that depend on the higher delay variables as parameters. An advantage of this consideration is that the analysis is made on a finite-dimensional system instead of an infinite-dimensional one. Finally, properties of the extended space, and the extension of the Lie bracket given in Definition 10 are used to define an algorithm that computes efficiently the extended Lie bracket operation. This operation is useful in the computation in applications as the ones enlisted in Table 1.

4.1 Expanded Dynamics of Nonlinear Time-Delay Systems

Consider the delay-differential system given by the equation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= f(\mathbf{x}(t), \mathbf{x}(t-1)), \\ \mathbf{x}(t) &= \vartheta_0(t), \qquad -1 \leq t \leq 0. \end{aligned} \tag{65}$$

The solution of the equation (65), computed by the step method, on the interval $(k-1) \leq t \leq k, k \in \mathbb{Z}^+$ is given by the differential equation

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}(t), \vartheta_{k-1}(t-1)), \\ \mathbf{x}(t) &= \vartheta_{k-1}(t), \qquad (k-2) \leq t \leq (k-1). \end{aligned} \tag{66}$$

The set of equations defined on the interval $(\alpha-1)\leq t\leq\alpha$

$$\begin{aligned}\dot{\mathbf{x}}(t) &= f(\mathbf{x}(t),\vartheta_{\alpha-1}(t-1)),\\ \dot{\mathbf{x}}(t-1) &= f(\mathbf{x}(t-1),\vartheta_{\alpha-2}(t-2)),\\ &\vdots\\ \dot{\mathbf{x}}(t-(\alpha-1)) &= f(\mathbf{x}(t-(\alpha-1)),\vartheta_{0}(t-\alpha)),\end{aligned}\tag{67}$$

defines the solution on the first α units of time of the equation (65) shifted into the interval $(\alpha-1)\leq t\leq\alpha$. This means that it is possible to group the solution of (65) on the interval $(\alpha-1)\leq t\leq\alpha$. Moreover, setting $t_0=(\alpha-1)$, The solution of (65) on $(\alpha-1)\leq t\leq\alpha$ can be computed solving the extended delay-differential equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= f(\mathbf{x}(t),\mathbf{x}(t-1)),\\ \dot{\mathbf{x}}(t-1) &= f(\mathbf{x}(t-1),\mathbf{x}(t-2)),\\ &\vdots\\ \dot{\mathbf{x}}(t-(\alpha-1)) &= f(\mathbf{x}(t-(\alpha-1)),\mathbf{x}(t-\alpha)),\\ \mathbf{x}(t) &= \vartheta_k(t), \qquad (k-1)\leq t\leq k, k\in[0,\alpha], \alpha\in\mathbb{Z}^{+},\end{aligned}\tag{68}$$

on the interval $t_0\leq t\leq t_0+1$.

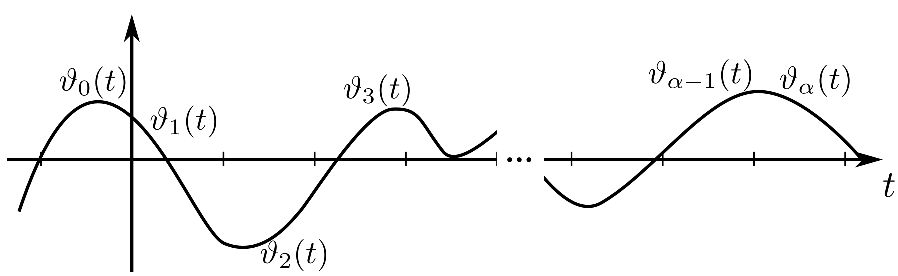


Figure 12: Solution for a time-delay system.

Since the extended system described by equation (68) is a system of delay-free differential equations, the conditions for existence and uniqueness of the solution, for the interval $0<t<\alpha$ are the same as the ones for non-autonomous differential equations. Moreover, through a change of variable $x_0(t)=x(t)$, $x_1(t)=x(t-1)$, $x_\alpha(t)=x(t-\alpha)$,

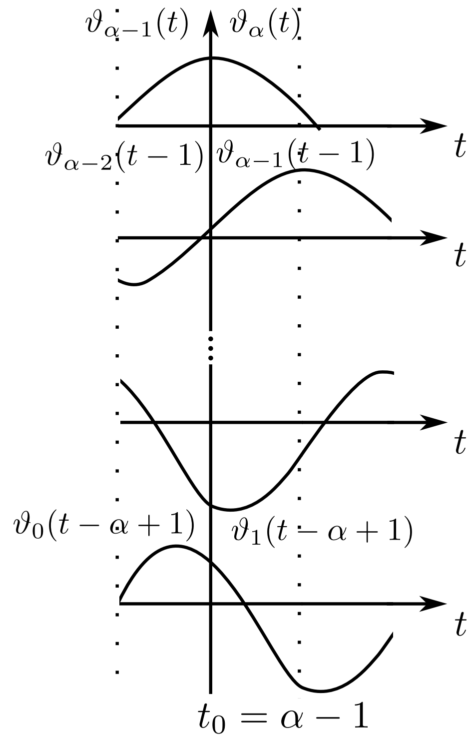


Figure 13: Solution for a extended time-delay system.

system (68) also can be represented as

$$\begin{aligned}
\dot{x}_1(t) &= f(x_0(t), x_1(t)), \\
\dot{x}_2(t) &= f(x_1(t), x_2(t)), \\
&\vdots \\
\dot{x}_{\alpha-1}(t) &= f(x_{\alpha-1}(t), \vartheta_0(t-1)),
\end{aligned}$$

with initial conditions $x_1(0) = x_1(1)$, $x_2(0) = x_3(1)$, $\ldots$, $x_{\alpha-1}(0) = \vartheta_0(0)$. The above analysis can be easily extended to the case of multiple delays.

Example 2 *Consider the time-delay system described by the differential equation*

$$\dot{x}(t) = -ax(t) + bx(t-1), \tag{69}$$

with $x(t) = 1$ for $t \in [-1, 0]$. Now, associate system

$$\dot{z}_0(t) = -az_0(t) + b\varphi(t) \tag{70}$$

to the system (69)*. System* (70) *is a copy of* (69) *considering* $x(t)$ *for* $t \in [-1, 0]$ *as an external signal* $\varphi(t+1)$ *with* $z_0(0) = x(0) = 1$*. This approximation is defined for a time window on the interval* $0 \leq t < 1$*. Computing the solution of* (70) *for the interval* $0 \leq t < 1$ *is exactly the first step of the step method to compute the solution of* (69) *on the interval* $0 \leq t < 1$*. This can be observed in Figure 14 which is the simulation of the system* (70) *compared with the numerical solution of* (69)*. Note that for* $t > 1$ *the solution of* (70) *does*

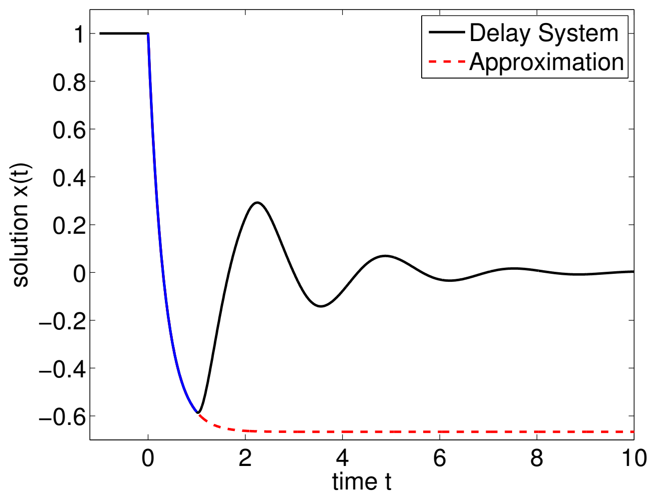


Figure 14: Extended system solution for $a = 3$**, and** $b = -2$ **for a window** $0 \leq t < 1$**.**

not match with the one of (69)*. The next approximation is endowed using an extended system of ordinary differential equations. Let us call* $z_1(t) = \varphi_1(t)$ *the solution of* (70) *on the interval* $0 \leq t < 1$*.*

Continuing with the step method, the solution of

$$\dot{x}(t) = -ax(t) + b\varphi_1(t-1), \tag{71}$$

for $1 \leq t < 2$*. Shifting* (71) *a units of time to the left we have*

$$\dot{x}(t+1) = -ax(t+1) + b\varphi_1(t), \tag{72}$$

is found for $0 \leq t < 1$*. Now setting the change of variable* $z_1(t) = x(t-1)$ *we are able to write the next extended system*

$$\begin{aligned} \dot{z}_0(t) &= -az_0(t) + bz_1(t) \\ \dot{z}_1(t) &= -az_1(t) + b\varphi(t) \end{aligned} \tag{73}$$

with $z_0(0) = z_2(1)$*,* $z_2(0) = 1$*. Note that doing this, the solution* $z_1(t) = \varphi_2(t)$ *on the interval* $0 \le t < 1$ *is equivalent to the solution of the second step of the step method shifted* 1 *unit of time to the left. So doing this, the solution of* (73) *defines the window approximation for* $0 \le t < 2$*. The solution of* $z_1(t)$ *on* $0 \le t < 1$ *defines the approximation for the section* $0 \le t < 1$ *while the solution of* $z_0(t)$ *on* $0 \le t < 1$ *describes the approximated system on* $1 \le t < 2$*. The same step method approach can be done for larger windows of time. The*

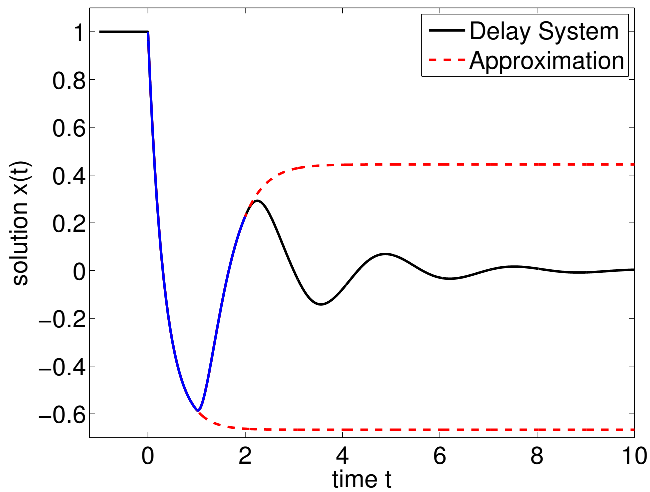


Figure 15: Extended system solution for $a = 3$, and $b = -2$ for a window $0 \le t < 2$.

approximation on the section $0 \le t < 3$ *is defined using the next set of equations*

$$\begin{aligned} \dot{z}_0(t) &= -az_0(t) + bz_1(t) \\ \dot{z}_1(t) &= -az_1(t) + bz_2(t) \\ \dot{z}_2(t) &= -az_2(t) + b\varphi(t) \end{aligned} \tag{74}$$

with $z_0(0) = z_1(1),\ z_1(0) = z_2(1),\ z_2(0) = 1$*. Where the solutions of* $z_1(t)$*,* $z_2(t)$*, and* $z_3(t)$ *on* $0 \le t < 3$ *defines the sections* $2 \le t < 3$*,* $1 \le t < 2$*, and* $0 \le t < 1$ *respectively. In the Figure 16 the approximation on the window* $0 \le t < 3$ *is represented by the solid blue line. The approximation can be extended until the* k*-th order given by the ordinary differential*

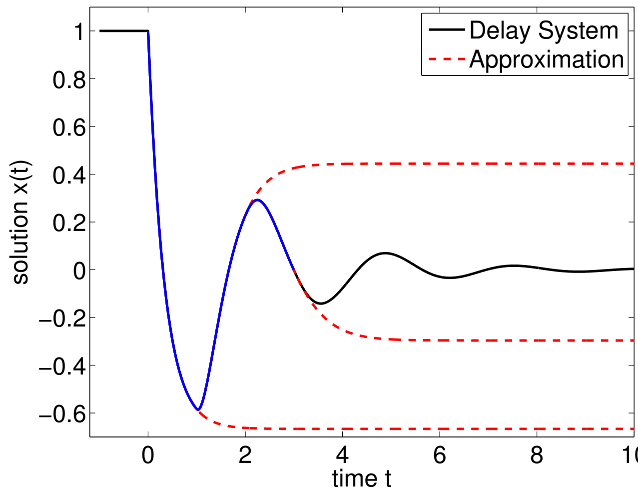


Figure 16: Extended system solution for $a = 3$, $b = -2$, and $1 = 1$ for a window $0 \leq t < 3$.

system

$$\begin{aligned} \dot{z}_0 &= az_0(t) + bz_1(t) \\ \dot{z}_1 &= az_1(t) + bz_2(t) \\ \dot{z}_2 &= az_2(t) + bz_3(t) \\ &\vdots \\ \dot{z}_k &= az_k(t) + b\varphi(t) \end{aligned} \tag{75}$$

with initial conditions $z_0(0) = z_1(1), z_1(0) = z_2(1), \ldots, z_k(0) = \varphi(0)$. *Where the solutions* $z_1(t), \ldots, z_2(t)$ *on* $0 \leq t < 1$ *defines the approximation on the sections* $k \leq k+1, \ldots, 0 \leq t < 1$ *respectively. Figure 17 shows the simulation of the approximation for a large value of* k. *The solution on the interval* $0 \leq t \leq 1$ *of the equation* (75) *is presented in Figure 18*

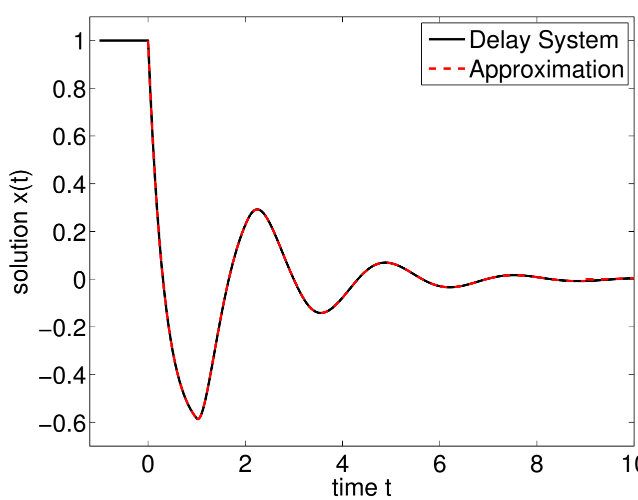


Figure 17: Extended system solution for $a = 3$, and $b = -2$ for a window $0 \leq t < 9$.

for $k = 9$. ◀

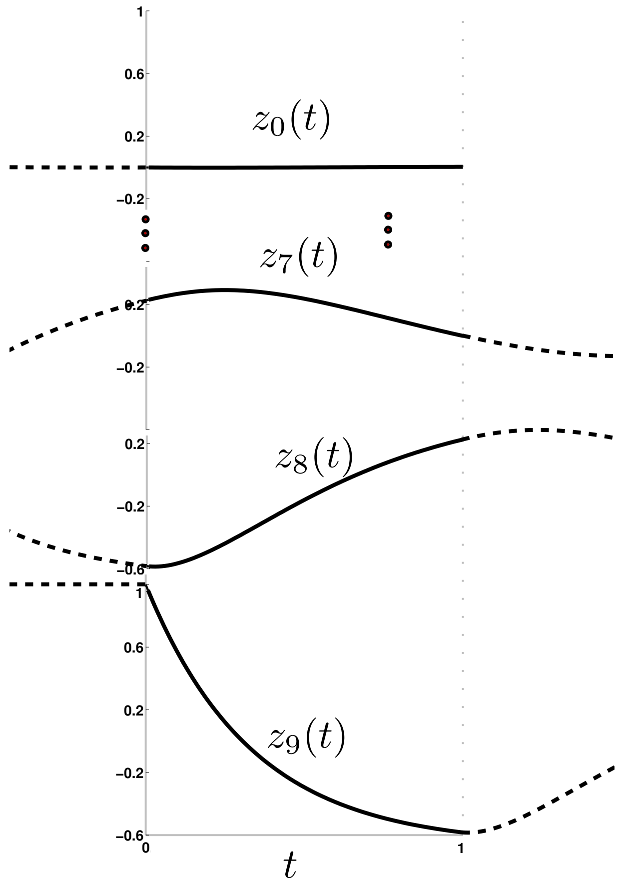


Figure 18: Solution for a extended time-delay system on the interval $0 \leq t \leq 10$**.**

4.2 Relative degree

As an application of the analysis included in the previous section, consider the nonlinear dynamical time-delay system with commensurable delays represented by the equation

$$\Sigma : \ \dot{x}(t) \ = \ F(x(t), x(t-1), \ldots, x(t-s)) + \sum_{j=0}^{s} G_j(x(t), x(t-1), \ldots, x(t-s))u(t-j), \tag{76}$$

and the dynamics described by the equation

$$\Sigma_0 : \ \dot{z}_0(t) \ = \ F(z_0(t), z_1, \ldots, z_s) + \sum_{j=0}^{s} G_j(z_0(t), z_1, \ldots, z_s)v_j \ , \tag{77}$$

where $v_0, v_1, \ldots, v_s$ are independent, and $z_1, \ldots, z_s$ are constant parameters.

Lemma 17 *Consider a function* $\phi(z_0(t))$*. Given the system* Σ*, the relative degree of*

$\phi(x(t))$ is greater or equal to 2 if and only if

$$d\phi(x(t)) \in \ker_{\mathcal{K}(\delta]}\{G_i(x(t), x(t-1), \ldots, x(t-s)), i = 0, \ldots, s\} \tag{78}$$

Proof. The time derivative of $\phi(x(t))$ is given by

$$\dot{\phi}(x(t)) \;=\; \frac{\partial \phi(x(t))}{\partial x(t)}\Big(F(x(t), \ldots, x(t-s)) + \textstyle\sum_{j=0}^{s} G_j(x(t), \ldots, x(t-s))u(t-j)\Big) \;, \tag{79}$$

Then, the derivative is independent of the input variable for $\frac{\partial \phi(x(t))}{\partial z_0(t)} G_j(x(t), \ldots, x(t-s)) = 0$ for $j = 0, \ldots, s$ that implies (78). ■

Lemma 18 *Given the system Σ_0, the relative degree of $\phi(z_0(t))$ is greater or equal to 2 if and only if*

$$d\phi(z_0(t)) \in \ker_{\mathcal{K}(\delta]}\{G_i(z_0(t), z_1, \ldots, z_s), i = 0, \ldots, s\} \tag{80}$$

Proof. The time derivative of $\phi(z_0(t))$ is given by

$$\dot{\phi}(z_0(t)) \;=\; \frac{\partial \phi(z_0(t))}{\partial z_0(t)}\Big(F(z_0(t), z_1, \ldots, z_s) + \textstyle\sum_{j=0}^{s} G_j(z_0(t), z_1, \ldots, z_s)v_j\Big), \tag{81}$$

Then $\frac{\partial \phi(z_0(t))}{\partial z_0(t)} G_j(z_0(t), z_1, \ldots, z_s) = 0$ for $j = 0, \ldots, s$ ■

From this result, and Lemma 81, the following corollary is stated

Corollary 19 *The relative degree of $\phi(z_0(t))$ for Σ_0 is greater or equal to 2, if and only if the relative degree of $\phi(x(t))$ for Σ is greater or equal to 2.*

Let us define the extended dynamical system Σ_s given by the set of equations

$$\Sigma_s : \begin{cases} \dot{z}_0(t) &= F(z_0(t), \ldots, z_s(t)) + \sum_{j=0}^{s} G_j(z_0(t) \ldots, z_s(t))v_j \\ \dot{z}_1(t) &= F(z_1(t), \ldots, z_s(t), z_{s+1}) + \sum_{j=0}^{s} G_j(z_0(t) \ldots, z_s(t), z_{s+1})v_{j+1} \\ \vdots & \\ \dot{z}_s(t) &= F((z_s(t), z_{s+1}, \ldots, z_{2s+1})) + \sum_{j=0}^{s} G_j(z_s(t), z_{s+1}, \ldots, z_{2s+1})v_{j+s} \end{cases}, \tag{82}$$

Corollary 20 *The relative degree of $\phi(z_0(t))$ for Σ_s is greater or equal to 3 if and only if the relative degree of $\phi(x(t))$ for Σ is greater or equal to 3.*

Proof. The proof comes from the Corollary 19. From the equation (81), and considering a relative degree larger than 2

$$\begin{aligned} \dot{\phi}(z_0(t)) &= \big(L_F\phi\big)(z_0(t), \ldots, z_s(t)) \\ \ddot{\phi}(z_0(t)) &= \sum_{i=0}^{s} \Big(\tfrac{\partial L_F\phi}{\partial z_i(t)}(z_0(t), \ldots, z_s(t))\Big) \cdot \dot{z}_i(t) \end{aligned} \tag{83}$$

and from (79)

$$\begin{aligned} \dot{\phi}(x(t)) &= \big(L_F\phi\big)(x(t), \ldots, x(t-s)) \\ \ddot{\phi}(x(t)) &= \sum_{i=0}^{s} \Big(\tfrac{\partial L_F\phi}{\partial x(t-s)}(x(t), \ldots, x(t-s))\Big) \cdot \dot{x}(t-i). \end{aligned} \tag{84}$$

From this it is shown that the conditions for the system Σ_s for having a relative degree greater or equal to 3 are necessary and sufficient for having a relative degree equal or greater than 3 for the system Σ ■

Now let us consider the extended system Σ_{sk}

$$\Sigma_{sk}:\begin{cases}\dot z_0(t) &= F(z_0(t),\ldots,z_s(t))+\sum_{j=0}^{s}G_j(z_0(t)\ldots,z_s(t))v_j\\ &\vdots\\ \dot z_{s(k-1)}(t) &= F(z_{s(k-1)}(t),\ldots,z_{sk}(t),z_{sk+1})+\\ & \quad\sum_{j=0}^{s}G_j(z_{s(k-1)}(t),\ldots,z_{sk}(t),z_{sk+1})v_{j+s(k-1)+1}\\ &\vdots\\ \dot z_{sk}(t) &= F((z_{sk}(t),z_{sk+1},\ldots,z_{2sk+1}))+\\ & \quad\sum_{j=0}^{s}G_j(z_{sk}(t),z_{sk+1},\ldots,z_{2sk+1})v_{j+ks+1}\end{cases}, \tag{85}$$

Corollary 21 *The relative degree of $\phi(z_0(t))$ for $\Sigma_s k$ is greater or equal to $k+2$, if and only if the relative degree of $\phi(x(t))$ for Σ is greater or equal to $k+2$.*

The proof for the Corollary 21 comes from the successive derivation of the output as it is done in the Corollary 20.
The same logic can be used with a function $\phi(z_0(t),\ldots,z_s(t))$.

Corollary 22 *The relative degree of $\phi(z_0(t),\ldots,z_s(t))$ for $\Sigma_s k$ is greater or equal to $k+2$, if and only if the relative degree of $\phi(x(t),\ldots,x(t-s))$ for Σ is greater or equal to $k+2$.*

4.3 Effective computation of the Extended Lie Bracket

In this section properties of the extended Lie bracket (16) are described, concerning the reduction of operations for the effective computation of this product operation on the elements of the distributions Δ_i, Δ_i', $i=0,\ldots,\gamma$ as defined in (23). An algorithm based on these properties that can be used in the solution of problems as integrability (Califano *et al.*, 2011a), and existence of a linearizing feedback (Califano and Moog, 2011), among others, is presented. Consider the following proposition reported in García-Ramírez and Márquez-Martínez (2014).

Proposition 23 *Let $\mathbf{r}_1(\mathbf{x},\mathbf{u},\delta)=\sum_{j=0}^{s}\mathbf{r}_1^j(\mathbf{x},\mathbf{u})\delta^j$, and $\mathbf{r}_2(\mathbf{x},\mathbf{u},\delta)=\sum_{j=0}^{s}\mathbf{r}_2^j(\mathbf{x},\mathbf{u})\delta^j$, and consider the set of operations defined by the extended Lie bracket $\varepsilon_j=[\mathbf{r}_1^j,\mathbf{r}_2^{q+j}]_{E_i},\ p\leq$*

$q, i \leq q,\ j = 0, \ldots, p$. *It is possible to compute* ε_j *using the next iterative expression*

$$\varepsilon_{j+1} = [\mathbf{r}_1^j, \mathbf{r}_2^{q+j}]_{E_0} \frac{\partial}{\partial x(t)} + \varepsilon_j \mid_{\mathbf{x}(-i)}, \quad \varepsilon_0 = 0 \tag{86}$$

Proof. It comes from the definition of the extended Lie bracket

$$\begin{aligned}
[\mathbf{r}_1^j(\cdot), \mathbf{r}_2^{q+j}(\cdot)]_{E_i} &= \sum_{\kappa=0}^{j} ([\mathbf{r}_1^{j-\kappa}(\cdot), \mathbf{r}_2^{q+j-\kappa}(\cdot)]_{E_0})^T \Big|_{(x(-\kappa))} \frac{\partial}{\partial x(t-\kappa)} \\
&= ([\mathbf{r}_1^j(\cdot), \mathbf{r}_2^{q+j}(\cdot)]_{E_0})^T \frac{\partial}{\partial x(t)} + \sum_{\kappa=1}^{j} ([\mathbf{r}_1^{j-\kappa}(\cdot), \mathbf{r}_2^{q+j-\kappa}(\cdot)]_{E_0})^T \Big|_{(x(-\kappa))} \frac{\partial}{\partial x(t-\kappa)} \\
&= ([\mathbf{r}_1^j(\cdot), \mathbf{r}_2^{q+j}(\cdot)]_{E_0})^T \frac{\partial}{\partial x(t)} + \sum_{\bar{\kappa}=0}^{j-1} ([\mathbf{r}_1^{j-\bar{\kappa}-1}(\cdot), \mathbf{r}_2^{q+j-\bar{\kappa}-1}(\cdot)]_{E_0})^T \Big|_{(x(-\bar{\kappa}-1))} \frac{\partial}{\partial x(t-\bar{\kappa}-1)} \\
&= ([\mathbf{r}_1^j(\cdot), \mathbf{r}_2^{q+j}(\cdot)]_{E_0})^T \frac{\partial}{\partial x(t)} + \varepsilon_j \mid_{\mathbf{x}(-i)}
\end{aligned}$$

where $\bar{\kappa} = \kappa - 1$, and $\varepsilon_j \mid_{\mathbf{x}(-i)}$ is $[\mathbf{r}_1^{j-1}(\cdot), \mathbf{r}_2^{q+j-1}(\cdot)]_{E_i}|_{(x(-1))}$ shifted by n rows. ■

Note that, instead of performing j operations in the form of equation (16), at each step time and row shifts substitutes a total of $p(p+1)/2$ symbolic derivative operations.

The proof of Proposition 23 can be visualized using the set of vectors

$$\left\{[\mathbf{r}_1^0, \mathbf{r}_2^\rho]_{E_i}, \ldots, [\mathbf{r}_1^{p-1}, \mathbf{r}_2^{q-1}]_{E_i},\ [\mathbf{r}_1^p, \mathbf{r}_2^q]_{E_i}\right\} =$$
$$\left\{ \begin{pmatrix} [\mathbf{r}_1^0, \mathbf{r}_2^\rho]_{E_0} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} [\mathbf{r}_1^1, \mathbf{r}_2^{\rho+1}]_{E_0} \\ [\mathbf{r}_1^0, \mathbf{r}_2^\rho]_{E_0}(-1) \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} [\mathbf{r}_1^{p-1}, \mathbf{r}_2^{q-1}]_{E_0} \\ [\mathbf{r}_1^{p-2}, \mathbf{r}_2^{q-2}]_{E_0}(-1) \\ [\mathbf{r}_1^{p-3}, \mathbf{r}_2^{q-3}]_{E_0}(-2) \\ \vdots \\ [\mathbf{r}_1^0, \mathbf{r}_2^\rho]_{E_0}(-p+1) \\ 0 \end{pmatrix}, \begin{pmatrix} [\mathbf{r}_1^p, \mathbf{r}_2^q]_{E_0} \\ [\mathbf{r}_1^{p-1}, \mathbf{r}_2^{q-1}]_{E_0}(-1) \\ [\mathbf{r}_1^{p-2}, \mathbf{r}_2^{q-2}]_{E_0}(-2) \\ \vdots \\ [\mathbf{r}_1^1, \mathbf{r}_2^{\rho+1}]_{E_0}(-p+1) \\ [\mathbf{r}_1^0, \mathbf{r}_2^\rho]_{E_0}(-p) \end{pmatrix} \right\} \tag{87}$$

with $\rho = q - p$. Consider now the following lemma.

Lemma 24 *Let* $\mathbf{r}_1(\mathbf{x}_{[\alpha]}, \delta) = \sum_{j=0}^{s} \mathbf{r}_1^j(\mathbf{x}_{[\boldsymbol{\alpha}]})\delta^j$, $\mathbf{r}_2(\mathbf{x}_{[\alpha]}, \delta) = \sum_{j=0}^{s} \mathbf{r}_2^j(\mathbf{x}_{[\boldsymbol{\alpha}]})\delta^j$*, and, without loss of generality,* $s = max(deg(\mathbf{r}_1(\cdot), \mathbf{r}_2(\cdot)), \alpha)$. *Then*

$$\left[\mathbf{r}_1^k(\cdot), \mathbf{r}_2^l(\cdot)\right]_{E_0} = 0 \tag{88}$$

for $k, l > s$.

Proof. It comes directly from equation (17) in Definition 10, (*i.e.* the definition of the extended Lie Bracket) since $\mathbf{r}_1^k(\cdot), \mathbf{r}_2^l(\cdot) = 0$ due to the condition established in the premise of the lemma $s = max(deg(\mathbf{r}_1(\cdot), \mathbf{r}_2(\cdot)), \alpha)$. ■

Lemma 24 implies that for $p > s$ in equation (87) it is not necessary to make any other computation than delay and row shifts in the computation of the extended Lie Bracket since the n elements of the resultant vector are zero. Furthermore, given finite dimensional distributions $\Delta_{\bar{\gamma}}$, $\Delta'_{\bar{\gamma}}$ given by (23), because of property P.ii, from Section 2.4, the extended Lie bracket

$$[\mathbf{r}_1^{\bar{k}}(\cdot), \mathbf{r}_2^{\bar{l}}(\cdot)]_{E_{\bar{\gamma}}} = 0, \tag{89}$$

for $\bar{k}, \bar{l} > s + \bar{\gamma}$ or $\mid \bar{k} - \bar{l} \mid > 2s$.

Another consideration for the computation of the extended Lie bracket can be pointed out by the following example.

Example 3 *Consider the right module defined as*

$$\Delta = span_{\mathcal{K}(\delta]} \left\{ \begin{pmatrix} 2x_1(t)x_1^2(t-1)\delta - 6x_1(t)\delta^2 \\ -x_2(t)x_1^2(t) + 3x_2(t)\delta \end{pmatrix} \right\}. \tag{90}$$

The involutivity of Δ'_i *distributions is now tested. The distribution*

$$\Delta'_0 \quad = \quad span_{\mathcal{K}} \left\{ -x_2(t)x_1^2(t)\tfrac{\partial}{\partial x_2(t)}, 2x_1(t)x_1^2(t-1)\tfrac{\partial}{\partial x_1(t)} + 3x_2(t)\tfrac{\partial}{\partial x_2(t)}, -6x_1(t)\tfrac{\partial}{\partial x_1(t)} \right\}$$

is involutive since its dimension is two and is generated by two independent vectors and it is not necessary to perform the extended Lie Bracket operation.

Example 3 shows that for Δ_k, $k = 0, 1, \ldots, \bar{\gamma}$, $\bar{\gamma} \in \mathbb{Z}^+$, can define a locally full rank distribution, and the computation of the extended Lie bracket is not needed.

Considering Proposition 23, Lemma 24, and Example 3 the following algorithm for the computation of the extended Lie bracket for two elements that belong to distribution Δ'_γ is presented

Algorithm 1

STEP 0.

Check: *Is $\Delta'_{\bar\gamma}$ generated by $\bar\gamma n$ independent vectors?*

YES: STOP, NO: Continue to next step.

STEP 1.

STEP 1.1 *Compute* $[\mathbf{r}_i^0(\cdot),\mathbf{r}_j^0(\cdot)]_{E_{\bar\gamma}} = \left(\mathbf{r}_i^0(\cdot),\mathbf{r}_j^0(\cdot)]_{E_0}\right)^T \frac{\partial}{\partial x(t)} + 0\frac{\partial}{\partial x(t-1)} + \ldots + 0\frac{\partial}{\partial x(t-\bar\gamma)}$ *Is*

$$[\mathbf{r}_i^j(\cdot),\mathbf{r}_j^j(\cdot)]_{E_{\bar\gamma}} \in \Delta'_{\bar\gamma}?$$

NO: STOP, YES: Continue to the next step

STEP 1.2 *Compute* $[\mathbf{r}_i^1(\cdot),\mathbf{r}_j^1(\cdot)]_{E_0}$*, and shift one unit of time* $[\mathbf{r}_1^0(\cdot),\mathbf{r}_2^0(\cdot)]_{E_0}$ *from STEP 1.0. Is*

$$\begin{aligned}[\mathbf{r}_i^1(\cdot),\mathbf{r}_j^1(\cdot)]_{E_{\bar\gamma}} = &\left(\mathbf{r}_i^1(\cdot),\mathbf{r}_j^1(\cdot)]_{E_0}\right)^T \frac{\partial}{\partial x(t)} + \left([\mathbf{r}_i^0(\cdot),\mathbf{r}_j^0(\cdot)]_{E_0}\right)^T\Big|_{\mathbf{x}(-1)} \frac{\partial}{\partial x(t-1)} + \\ &0\frac{\partial}{\partial x(t-2)} + \ldots + 0\frac{\partial}{\partial x(t-\bar\gamma)} \\ &\in \Delta'_{\bar\gamma}?\end{aligned}$$

NO: STOP, YES: Continue to the next step

STEP 1.k *Compute* $[\mathbf{r}_i^{k-1}(\cdot),\mathbf{r}_j^{k-1}(\cdot)]_{E_0}$*, with* $[\mathbf{r}_i^{k-1}(\cdot),\mathbf{r}_j^{k-1}(\cdot)]_{E_0} = 0$ *for* $k-1 < s$*, and shift* $\bar k$ *units of time* $[\mathbf{r}_1^{\bar k}(\cdot),\mathbf{r}_2^{\bar k}(\cdot)]_{E_0}$ *from STEP 1.$\bar k$, for* $\bar k = 0,\ldots,k-2$*. Is*

$$\begin{aligned}[\mathbf{r}_i^{k-1}(\cdot),\mathbf{r}_j^{k-1}(\cdot)]_{E_{\bar\gamma}} = &\left(\mathbf{r}_i^{k-1}(\cdot),\mathbf{r}_j^{k-1}(\cdot)]_{E_0}\right)^T \frac{\partial}{\partial x(t)} + \sum_{\bar k=0}^{k-3}\left([\mathbf{r}_i^{\bar k}(\cdot),\mathbf{r}_j^{\bar k}(\cdot)]_{E_0}\right)^T\Big|_{\mathbf{x}(-k+\bar k+2)} \frac{\partial}{\partial x(t-k+\bar k+2)} + \\ &0\frac{\partial}{\partial x(t-k+1)} + \ldots + 0\frac{\partial}{\partial x(t-\bar\gamma)} \\ &\in \Delta'_{\bar\gamma}?\end{aligned}$$

NO: STOP, YES: Continue to the next step until $k > s + \bar\gamma$

STEP ***ρ.k*** *Compute* $[\mathbf{r}_i^{k-1}(\cdot),\mathbf{r}_j^{k+\rho-1}(\cdot)]_{E_0}$*, with* $[\mathbf{r}-\mathbf{1}_i^k(\cdot),\mathbf{r}_j^{k+\rho-1}(\cdot)]_{E_0} = 0$ *for* $k-1 > s$*, and* *shift $\bar{k}$ unit of time $[\mathbf{r}_1^{\bar{k}}(\cdot),\mathbf{r}_2^{\bar{k}+\rho}(\cdot)]_{E_0}$ from STEP 1.$\bar{k}$, for $\bar{k}=0,\ldots,k-2$. Is*

$$\begin{aligned}[\mathbf{r}_i^{k-1}(\cdot),\mathbf{r}_j^{k+\rho-1}(\cdot)]_{E_{\bar{\gamma}}} =& \left(\mathbf{r}_i^{k-1}(\cdot),\mathbf{r}_j^{k+\rho-1}(\cdot)]_{E_0}\right)^T \frac{\partial}{\partial x(t)}+\\ &\sum_{\bar{k}=0}^{k-3}\left(\mathbf{r}_i^{\bar{k}}(\cdot),\mathbf{r}_j^{\bar{k}+\rho}(\cdot)]_{E_0}\right)^T\Big|_{\mathbf{x}(-k+\bar{k}+2)}\frac{\partial}{\partial x(t-k+\bar{k}+2)}+\\ &0\frac{\partial}{\partial x(t-k+1)}+\ldots+0\frac{\partial}{\partial x(t-\bar{\gamma})}\\ \in\ & \Delta'_{\bar{\gamma}}?\end{aligned}$$

NO: STOP, YES: Continue to the next step

Continue answering until $k>s+\bar{\gamma}$ or $\rho>2s$. ♦

The following examples show how the Algorithm 1 is employed to compute the extended Lie Bracket on elements of Δ.

Example 4 *Consider the distribution defined by the submodule* (90) *from Example 3,*

$$\begin{aligned}\Delta'_1 &= span_{\mathcal{K}}\left\{\begin{pmatrix}0\\-x_2(t)x_1^2(t)\\0\\0\end{pmatrix},\begin{pmatrix}2x_1(t)x_1^2(t-1)\\3x_2(t)\\0\\-x_2(t-1)x_1^2(t-1)\end{pmatrix},\begin{pmatrix}-6x_1(t)\\0\\2x_1(t-1)x_1^2(t-2)\\3x_2(t-1)\end{pmatrix},\begin{pmatrix}0\\0\\-6x_1(t-1)\\0\end{pmatrix}\right\}\\ &= span_{\mathcal{K}}\Big\{-x_2(t)x_1^2(t)\frac{\partial}{\partial x_2(t)}, 2x_1(t)x_1^2(t-1)\frac{\partial}{\partial x_1(t)}+3x_2(t)\frac{\partial}{\partial x_2(t)}-\\ &\quad x_2(t-1)x_1^2(t-1)\frac{\partial}{\partial x_2(t-1)}, -6x_1(t)\frac{\partial}{\partial x_1(t)}+2x_1(t-1)x_1^2(t-2)\frac{\partial}{\partial x_1(t-1)}+\\ &\quad 3x_2(t-1)\frac{\partial}{\partial x_2(t-1)}, -6x_1(t-1)\frac{\partial}{\partial x_1(t-1)}\Big\}\\ &= span_{\mathcal{K}}\left\{\mathbf{r}_1^0(\mathbf{x}_{[\mathbf{2}]}),\mathbf{r}_1^1(\mathbf{x}_{[\mathbf{2}]}),\mathbf{r}_1^2(\mathbf{x}_{[\mathbf{2}]}),\mathbf{r}_1^3(\mathbf{x}_{[\mathbf{2}]})\right\}.\end{aligned}$$

STEP 1.1. *The extended Lie bracket $\left[\mathbf{r}_1^0(\mathbf{x}_{[\mathbf{2}]}),\mathbf{r}_1^1(\mathbf{x}_{[\mathbf{2}]})\right]_{E_1}$ is computed*

$$\begin{aligned}\left[\mathbf{r}_1^0(\mathbf{x}_{[\mathbf{2}]}),\mathbf{r}_1^1(\mathbf{x}_{[\mathbf{2}]})\right]_{E_0} &= \frac{\partial\mathbf{r}_1^1(\mathbf{x}_{[\mathbf{2}]})}{\partial\mathbf{x}(t)}\mathbf{r}_1^0(\mathbf{x}_{[\mathbf{2}]})-\frac{\partial\mathbf{r}_1^0(\mathbf{x}_{[\mathbf{2}]})}{\partial\mathbf{x}(t)}\mathbf{r}_1^1(\mathbf{x}_{[\mathbf{2}]})\\ &= \begin{pmatrix}2x_1^2(t-1) & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}0\\-x_2(t)x_1^2(t)\end{pmatrix}-\begin{pmatrix}0 & 0\\-2x_2(t)x_1(t) & -x_1^2(t)\end{pmatrix}\begin{pmatrix}2x_1(t)x_1^2(t-1)\\3x_2(t)\end{pmatrix}\\ &= \begin{pmatrix}0\\-4x_2(t)x_1^2(t)x_1^2(t-1)\end{pmatrix}.\end{aligned}$$

Setting $\left[\mathbf{r}_1^0(\mathbf{x}_{[\mathbf{2}]}),\mathbf{r}_1^1(\mathbf{x}_{[\mathbf{2}]})\right]_{E_1}=-4x_2(t)x_1^2(t)x_1^2(t-1)\frac{\partial}{\partial x_2(t)}\in\Delta'_1$.
All the other operations in STEP 1 are zero.

STEP 2.1.

$$\begin{aligned} \left[\mathbf{r}_1^0(\mathbf{x}_{[\mathbf{2}]}), \mathbf{r}_1^2(\mathbf{x}_{[\mathbf{2}]})\right]_{E_0} &= 12x_2(t)x_1^2(t)\tfrac{\partial}{\partial x_1(t)} \\ \left[\mathbf{r}_1^0(\mathbf{x}_{[\mathbf{2}]}), \mathbf{r}_1^2(\mathbf{x}_{[\mathbf{2}]})\right]_{E_1} &= 12x_2(t)x_1^2(t)\tfrac{\partial}{\partial x_2(t)} \in \Delta_1' \end{aligned} \tag{91}$$

STEP 2.2. *from the STEP 2.1*

$$\begin{aligned} \left[\mathbf{r}_1^1(\mathbf{x}_{[\mathbf{2}]}), \mathbf{r}_1^3(\mathbf{x}_{[\mathbf{2}]})\right]_{E_0} &= -6x_1(t)x_1^2(t-1)\tfrac{\partial}{\partial x_1(t)} \\ \left[\mathbf{r}_1^1(\mathbf{x}_{[\mathbf{2}]}), \mathbf{r}_1^3(\mathbf{x}_{[\mathbf{2}]})\right]_{E_0} &= -24x_1(t)x_1^2(t-1)\tfrac{\partial}{\partial x_1(t)} + 12x_2(t-1)x_1^2(t-1)\tfrac{\partial}{\partial x_1(t-1)} \\ &\in \Delta_1' \end{aligned} \tag{92}$$

The other elements in the following steps are zero.
In fact the function $x_1(t)x_2^2(t-1)$ fulfills the integrability condition given in Califano et al. *(2011a) since $x_2^2(t-1)dx_1(t) + 2x_1(t)x_2(t-1)\delta dx_2(t)$ defines the kernel of Δ.*

4.3.1 Numerical results.

In this section, numerical results are presented to compare step 0 from the proposed Algorithm 1 with the direct algorithmic computation of the extended Lie bracket. The algorithm used for the test consists in computing the extended Lie brackets

$$\left[r_1^j(\mathbf{x}_{[\mathbf{s}]}), r_2^j(\mathbf{x}_{[\mathbf{s}]})\right]_{E_i} \tag{93}$$

for $j = 1, \ldots, 12,\ i = 1, 3, 6, 9,\ r_1^j(\mathbf{x}_{[\mathbf{s}]}), r_2^j(\mathbf{x}_{[\mathbf{s}]}) \in \mathcal{K}^n(\delta]$. Different values of dimension of the polynomial vectors, polynomial degree on the delay operator, and degree on the state variable product (*i.e.* coefficient degree) were tested. Vectors were randomly generated. Figure 19 shows numerical results of the test for vectors generated in $\mathcal{K}^2(\delta]$, with polynomial degree 6 on δ, changing the degree of the coefficients of the polynomial vectors, which are generated by the product of the state variables.

Figure 20 displays results for different elements of $\mathcal{K}^2(\delta]$ with coefficient degree 6, for different polynomial degrees.

Finally, the test was performed varying the vector dimension from 2 to 6, keeping constant the values of the polynomial degree on δ as 6, and a coefficient degree as 1. Nu-

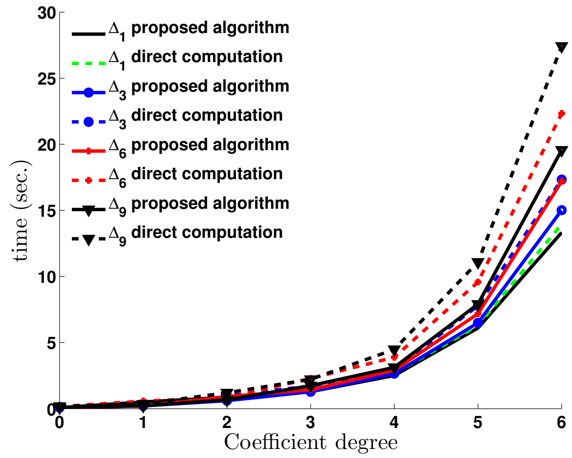


Figure 19: Numerical test of the proposed algorithm for different coefficient degrees.

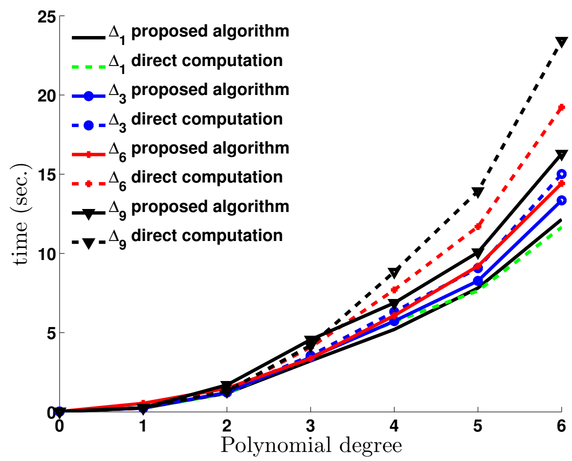


Figure 20: Numerical test of the proposed algorithm for different polynomial degrees.

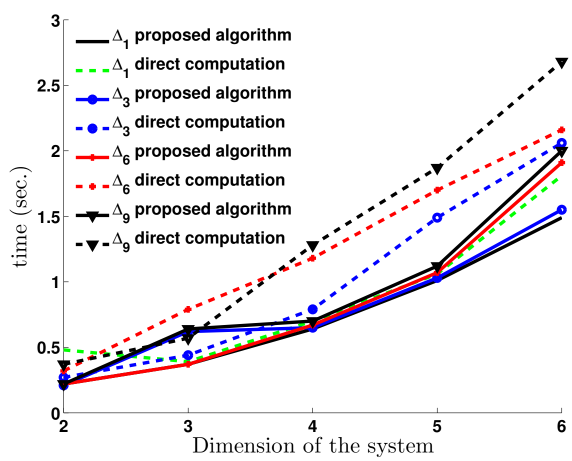


Figure 21: Numerical test of the proposed algorithm for different vector dimensions.

merical results are presented in Figure 21. It is important to remark that the saved time is notable for higher dimension, polynomial degree, and coefficient degree values. Note that, Algorithm 1 computes ρ-times a similar procedure. Computations in this section were performed using wxMaxima version 12.04.0 with a Maxima version 5.27.0, and GNU Common Lisp (GCL) version 2.6.7.

4.4 Discussion.

Contributions for a better understanding of the new geometrical approach for hereditary systems have been presented in this chapter. The algorithm presented in Section 4.3 is defined in such a way that can be implemented in computational software (see for example Gárate-García *et al.* (2011)) to simplify the cumbersome procedures that imply the big number of operations required by the conditions for solutions of problems as the ones enlisted in Table 1. The implementation of these algorithms directly benefits to the final user, scientists, and engineers, since this allows them to take advantage of the new results on geometrical control theory for time-delay system, with a reduced inversion of training and computational time. In Section 4.3.1 numerical results show the computational effectiveness of the proposed procedure.

Chapter 5. Linearization Via Input-Output Injection

In this chapter, it is discussed the equivalence of nonlinear system representation 2

$$\begin{aligned}\dot{\mathbf{x}}(t) &= f(\mathbf{x}_{[s]}) + \sum_{j=0}^{s} g_j(\mathbf{x}_{[s]})\mathbf{u}(t-j)\\ \mathbf{y}(t) &= h(\mathbf{x}_{[s]}),\end{aligned}$$

and the input-output representation

$$\psi(\mathbf{y}_{[s]}^{(n)}, \mathbf{y}_{[s]}^{(n-1)}, \ldots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-1)}, \ldots, \mathbf{u}_{[s]}) = 0 \tag{94}$$

to a canonical representation composed by a not necessarily delay-free linear part, and a nonlinear part, also known as injection function, that only depends on the input and the output, described by the equation

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \sum_{i=0}^{s} A_i\mathbf{z}(t-i) + \varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}),\\ y(t) &= \sum_{j=0}^{s} C_j\mathbf{z}(t-j).\end{aligned} \tag{95}$$

where $\mathbf{z} \in \mathbb{R}^n,\ u \in \mathbb{R},\ y \in \mathbb{R},\ A_i \in \mathbb{R}^{n\times n}$ for $i = 0, \ldots, s$, and $C_j \in \mathbb{R}^{1\times n}$ for $j = 0, \ldots, s$. The results presented in García-Ramírez *et al.* (2016) concerning constructive necessary and sufficient conditions to find a delay-free linear representation, modulo input-output injections, of an input-output nonlinear time-delay system representation are examined. Also the equivalence of dynamics (2) to an equivalent linearized, modulo input-output injection, system through an invertible change of coordinates are studied and used for the observer design problem. Also, the stronger condition of the existence of a bicausal transformation to this kind of linearized form is addressed. The main interest of the canonical form (95) is due to its viability in the design of a Luenberger-type observer. This is done by defining an observer dynamics for the system 95 is characterized by the equation

$$\dot{\zeta}(t) = \sum_{i=0}^{s} A_i\zeta(t-i) + \varphi(\mathbf{y}_{[\mathbf{s}]}, \mathbf{u}_{[\mathbf{s}]}) + \sum_{i=0}^{\bar{s}}\sum_{j=0}^{s} K_iC_j\zeta(t-i-j) - \sum_{i=0}^{s} K_iy(t-i). \tag{96}$$

For such dynamical system, an error equation given by

$$\mathbf{e}(t) = \mathbf{z}(t) - \zeta(t), \tag{97}$$

defines an error dynamics expressed as

$$\dot{\mathbf{e}}(t) = \sum_{i=0}^{s+\bar{s}} \left(A_i + \sum_{j=0}^{i} K_j C_{i-j} \right) \mathbf{e}(t-i). \tag{98}$$

It is important to consider that, while the solution of the differential equation defined by (96) converges asymptotically to the instantaneous state value of the linearized system (95), the computed information, corresponding to the nonlinear system (2), by the use of a non-bicausal invertible change of coordinates is time-delayed. Consequently, the bicausality of the transformation used is of interest in the pursuit of the desired canonical form.
The following problem is stated starting from the input-output representation:

Problem 1 *Given the input-output equation 94 where the differential ideal generated by* ψ *is prime, find, if possible, a realization of the form* (95)

Following problems are considered starting from the state-space representation of the system:

Problem 2 *Given the observable time-delay dynamical system* (2) *find, if possible, an invertible change of coordinates* $\mathbf{z}(t) = \phi(\mathbf{x}_{[p,s]})$ *such that* (2) *is transformed into* (95)*.*

Problem 3 *Given the observable time-delay dynamical system* (2) *find, if possible, a bicausal change of coordinates* $\mathbf{z}(t) = \phi(\mathbf{x}_{[s]})$ *such that* (2) *is transformed into* (95)*.*

The next section concerns the solution of Problem 1, which is important for the construction of the algorithm used to solve Problems 2, and 3.

5.1 Linear Delay-Free Realization for the Input-Output Representation.

In this section, the solution of Problem 1 is studied as it was presented in García-Ramírez *et al.* (2016). This is done as a preliminary for the solution of Problems 2, and

3. First, a lemma that concerns the three main problems of this chapter is presented as follows.

Lemma 25 *Problems 1, 2, and 3 are solvable, only if the given system admits an input-output equation of retarded type, and of the form*

$$y^{(n)}(t) = \sum_{i=1}^{n} \Phi_i(\mathbf{y}_{[\mathbf{s}]}, \mathbf{u}_{[\mathbf{s}]})^{(i-1)}. \tag{99}$$

Proof. Consider the differential form of (95) which is given by

$$\begin{aligned} d\dot{z}(t) &= A(\delta)dz(t) + d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) \\ dy(t) &= C(\delta)dz(t). \end{aligned} \tag{100}$$

Consider the differential form for the first derivative of the output given by

$$d\dot{y} = C(\delta) \cdot \big(A(\delta)dz(t) + d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})\big) \tag{101}$$

and iteratively

$$dy^{(k)} = C(\delta)A^k(\delta) \cdot dz(t) + \sum_{i=0}^{k-1} C(\delta)A^i(\delta)d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})^{(k-i-1)} \tag{102}$$

with $k = 0, \ldots n$. Because of the commutativity of $\mathbb{R}[\delta]$, the use of the Cayley-Hamilton's theorem is allowed, so it is possible to find $\sigma_i \in \mathbb{R}[\delta],\ i = 0, \ldots n$ such that

$$\sum_{i=0}^{n} \sigma_i C(\delta)A^i(\delta) = 0, \tag{103}$$

with $\sigma_n = 1$. Then the differential form of the input-output representation of (95) can be written as

$$dy^{(n)} = -\sum_{i=0}^{n-1} \sigma_i dy^{(i)}(t) + \sum_{k=1}^{n} \sum_{i=0}^{k-1} \sigma_k C(\delta)A^i(\delta)d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})^{(k-i-1)}, \tag{104}$$

which has the structure (104), with

$$d\Phi_{k+1}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) = -\sigma_k dy(t) + \sum_{j=0}^{n-k-1} \sigma_{n-j} C(\delta) A^{n-k-j-1}(\delta) d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$$

for all $k = 0, \ldots, n-1$. ■

In Márquez-Martínez *et al.* (2002), an observable system that may be written in the form (104) is called linearizable by additive output injections. An algorithm for the effective computation of the functions Φ_i was included. For the sake of completeness, that linearization algorithm is presented.

Define

$$\begin{aligned} E^0 &= 0 \\ E^k &= span_{\mathcal{K}(\delta]}\{dy(t), \ldots, dy^{(k-1)}(t), du(t), \ldots, du^{(k-1)}\} \end{aligned}$$

and assume that $dim_{\mathcal{K}(\delta]} E^n = 2n$.

Algorithm 2

STEP 0: Set $\psi_1 = \psi$, and compute the differential form of equation (94)

$$dy^{(n)}(t) = d\left(\psi_1(\mathbf{y}_{[s]}^{(n-1)}, \ldots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-1)}, \ldots, \mathbf{u}_{[s]})(t)\right) \tag{105}$$

STEP 1: By assumption $dy^{(n)} \in E^n$. Compute $\lambda_{n-1}^0 = \sum_{i=0}^{s} \frac{\partial \psi_1(\cdot)}{\partial y(t-i)^{(n-1)}} \delta^i$ (the coefficient of $dy^{(n-1)}(t)$) and $\mu_{n-1}^0 = \sum_{i=0}^{s} \frac{\partial \psi_1(\cdot)}{\partial u(t-i)^{(n-1)}} \delta^i$ (the coefficient of $du^{(n-1)}(t)$). Now Set

$$\omega_1 := \lambda_{n-1}^0 dy + \mu_{n-1}^0 du,$$

if $d\omega_1 \neq 0$ then STOP! there is no solution,

Compute $\Phi_1(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$ such that $\omega_1 = d\Phi_1(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$. Set

$$\begin{aligned} \psi_2(\mathbf{y}_{[s]}^{(n-2)}, \ldots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-2)}, \ldots, \mathbf{u}_{[s]}) \quad := \quad & \psi_1(\mathbf{y}_{[s]}^{(n-1)}, \ldots, \mathbf{y}_{[s]}, \mathbf{u}_{[s]}^{(n-1)}, \ldots, \mathbf{u}_{[s]}) - \\ & \Phi_1^{(n-1)}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}), \end{aligned}$$

Compute the differential form of equation $\psi_2(\cdot)$

$$d\left(\psi_2(\mathbf{y}_{[s]}^{(n-2)},\ldots,\mathbf{y}_{[s]},\mathbf{u}_{[s]}^{(n-2)},\ldots,\mathbf{u}_{[s]})\right) = d\left(\psi_1(\mathbf{y}_{[s]}^{(n-1)},\ldots,\mathbf{y}_{[s]},\mathbf{u}_{[s]}^{(n-1)},\ldots,\mathbf{u}_{[s]}) - \Phi_1^{(n-1)}(\mathbf{y}_{[s]},\mathbf{u}_{[s]})\right) \tag{106}$$

Check: $d\psi_2(t) \in E^{n-1}$*?*

NO: Stop, YES: Continue to the next step

STEP k*: Define* $\lambda_{n-k}^{k-1} = \sum_{i=0}^{s} \frac{\partial\psi_k(\cdot)}{\partial y(t-i)^{(n-k)}}\delta^i$ *and* $\mu_{n-k}^{k-1} = \sum_{i=0}^{s} \frac{\partial\psi_k(\cdot)}{\partial u(t-i)^{(n-k)}}\delta^i$ *as the coefficient of* $du^{(n-k)}(t)$ *from the last equation in step* $k-1$*. Now Set*

$$\omega_k := \lambda_{n-k}^{k-1}dy + \mu_{n-k}^{k-1}du,$$

if $d\omega_k \neq 0$ *then STOP! there is no solution,*

if $d\omega_k = 0$ *then compute* $\Phi_k(\mathbf{y}_{[s]},\mathbf{u}_{[s]})$ *such that* $\omega_k = d\Phi_k(\mathbf{y}_{[s]},\mathbf{u}_{[s]})$*, and set*

$$\psi_{k+1}(\mathbf{y}_{[s]}^{(n-k+1)},\ldots,\mathbf{y}_{[s]},\mathbf{u}_{[s]}^{(n-k+1)},\ldots,\mathbf{u}_{[s]}) := \psi_k(\mathbf{y}_{[s]}^{(n-k)},\ldots,\mathbf{y}_{[s]},\mathbf{u}_{[s]}^{(n-k)},\ldots,\mathbf{u}_{[s]}) - \Phi_k^{(n-k)}(\mathbf{y}_{[s]},\mathbf{u}_{[s]})$$

$$d\left(\psi_{k+1}(\mathbf{y}_{[s]}^{(n-k+1)},\ldots,\mathbf{y}_{[s]},\mathbf{u}_{[s]}^{(n-k+1)},\ldots,\mathbf{u}_{[s]})\right) = d\left(\psi_k(\mathbf{y}_{[s]}^{(n-k)},\ldots,\mathbf{y}_{[s]},\mathbf{u}_{[s]}^{(n-k)},\ldots,\mathbf{u}_{[s]}) - \Phi_k^{(n-k)}(\mathbf{y}_{[s]},\mathbf{u}_{[s]})\right) \tag{107}$$

Check: $d\psi_{k+1}(t) \in E^{n-k}$*?*

For $k = 2,\ldots,n$*.* ♦

If Algorithm 2 can be completed for each step k, for $k = 1,\ldots,n$, then it is possible to establish necessary and sufficient conditions for the solution of Problem 1, as it is stated in the next theorem

Theorem 26 *(García-Ramírez* et al.*, 2016) The input-output equation* (94) *admits a linear state-space representation up to input-output injection of the form* (95) *if and only if*

i) The system can be represented by an input-output equation of retarded type.

ii) The linearization Algorithm 2 ends with n exact one-forms ω_i

Then the state-space representation is obtained by setting

$$\begin{aligned} z_1(t) &= y(t) \\ z_2(t) &= \dot{y}(t) - \Phi_1(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) \\ &\vdots \\ z_{n-1}(t) &= y^{n-1}(t) - \sum_{i=1}^{n-1} \Phi_i^{(n-i-1)}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) \end{aligned} \tag{108}$$

Proof. Since the procedure is constructive, we only need to prove the necessity. To this end, recall that if the given system represented through its input-output equation can be written in the form (95) then, due to Lemma 25, necessarily the system must admit an input-output equation of retarded type given by

$$dy^{(n)} = -\sum_{i=0}^{n-1} \sigma_i dy^{(i)}(t) + \sum_{k=1}^{n} \sum_{i=0}^{k-1} \sigma_k C(\delta) A^i(\delta) d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})^{(k-i-1)}$$

which proves the necessity of i). Applying the linearization algorithm, one gets that at the generic step $k \leq n-1$

$$\omega_k = d\Phi_{k+1}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) = -\sigma_k dy(t) + \sum_{j=0}^{n-k-1} \sigma_{n-j} C(\delta) A^{n-k-j-1}(\delta) d\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})$$

which shows that the algorithm ends up with n exact differentials ω_k, that is, ii) must be satisfied. From (124), one thus gets that the state-space representation is given by

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) + \Phi_1(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}), \\ \dot{z}_2(t) &= z_3(t) + \Phi_2(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}), \\ &\vdots \\ \dot{z}_n(t) &= \Phi_n(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}), \\ y(t) &= z_1(t), \end{aligned} \tag{109}$$

which ends the proof. ■

As a corollary, one gets the following result.

Corollary 27 *If the input-output equation* (94) *admits a retarded linear state-space representation, up to input-output injection, then it can be written in the form* (109)*.*

Example 5 *This example illustrates the use of Algorithm 2 in the solution of Problem 1. Let us consider the input-output equation*

$$\ddot{y}(t) = y(t-1)u(t-3) + y(t)\dot{y}(t-1) + \dot{y}(t)y(t-1) \tag{110}$$

$$\begin{aligned} d\ddot{y}(t) &= y(t-1)du(t-3) + u(t-3)dy(t-1) + y(t)d\dot{y}(t-1) + \\ &\quad \dot{y}(t-1)dy(t) + \dot{y}(t)dy(t-1) + y(t-1)d\dot{y}(t) \end{aligned}$$

Following the algorithm, we define

$$\begin{aligned} \omega_1 &= y(t)dy(t-1) + y(t-1)dy(t) = d(y(t)y(t-1)) \\ \omega_2 &= u(t-3)dy(t-1) + y(t-1)du(t-3) = d(y(t-1)u(t-3)), \end{aligned}$$

which defines

$$\begin{aligned} z_1(t) &= y(t), \\ z_2(t) &= \dot{y}(t) - y(t)y(t-1), \end{aligned}$$

and then a realization of the equation (110) *is*

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) + y(t)y(t-1) \\ \dot{z}_2(t) &= y(t-1)u(t-3) \\ y(t) &= z_1(t) \end{aligned} \tag{111}$$

which is in the form (95)*.* ◁

Example 6 *Consider the system*

$$\ddot{y}(t) = -a_0 y(t) - a_1 y(t-3) + y(t-3)y(t-4) + u(t-3), \tag{112}$$

with a differential form given by

$$d\ddot{y}(t) = -a_0 dy(t) - a_1 dy(t-3) + y(t-3)dy(t-4) + y(t-4)dy(t-3) + du(t-3)$$

with the injection functions defined by

$$\begin{aligned} \omega_1 &= 0, \\ \omega_2 &= -a_0 dy(t) - a_1 dy(t-3) + y(t-3)dy(t-4) + y(t-4)dy(t-3) + du(t-3), \\ d\Phi_2 &= d\big(-a_0 y(t) - a_1 y(t-3) + y(t-3)y(t-4) + u(t-3)\big), \end{aligned}$$

that allows to define the change of variable

$$\begin{aligned} z_1(t) &= y(t), \\ z_2(t) &= \dot{y}(t), \end{aligned}$$

taking the system (112) *into the form*

$$\begin{aligned} \dot{z}_1(t) &= z_2(t), \\ \dot{z}_2(t) &= -a_0 z_1(t) - a_1 z_1(t-3) + z_1(t-3) z_1(t-4) + u(t-3), \\ y(t) &= z_1(t), \end{aligned} \tag{113}$$

which is in the desired form. Consider now the change of variables

$$\begin{aligned} z_1(t-3) &= y(t), \\ z_2(t-3) &= \dot{y}(t), \end{aligned}$$

$$\begin{aligned} \dot{z}_1(t) &= z_2(t), \\ \dot{z}_2(t) &= -a_0 z_1(t) - a_1 z_1(t-3) + z_1(t-3)z_1(t-4) + u(t), \\ y(t) &= z_1(t-3), \end{aligned} \tag{114}$$

Which is a different realization of (112) *not related by a bicausal change of coordinates with* (113)*. Note that* $\Phi_2 = -a_0 y(t+3) - a_1 y(t-3) + y(t-3)y(t-4) + u(t-3)$ *is no longer the injection function since is a noncausal equation.*

5.2 Association of the Input-Output Equation and the Linear Time-Delay System State-Space Representation

The objective of this section is to present several properties that relates the input-output representation and the state-space linear, up to input-output injection, time-delay system. The importance of the study of this relationship lies in the constructive conditions presented in García-Ramírez *et al.* (2016) for the solution of Problem 2, and by extension to the solution of Problem 3. As part of the linearizing procedure, the original nonlinear time-delay system is taken into the input-output representation to compute the injection functions of the desired normal form. To be able to do this it is needed to consider cases as the one that follows.

Example 7 *Consider the system*

$$\begin{aligned} \dot{x}(t) &= x(t)u(t) \\ y(t) &= x(t) + x(t-1), \end{aligned}$$

that has an input-output representation of neutral-type

$$\dot{y} - \alpha(\mathbf{u}_{[s]})\dot{y}(t-1) - u(t)y(t) + \alpha(\mathbf{u}_{[s]})u(t-2)y(t-1) = 0$$

where $\alpha(\mathbf{u}_{[s]}) = \frac{u(t-1)-u(t)}{u(t-1)-u(t-2)}$.

Example 7 shows that it is possible that a state-space representation of a time-delay system may have a neutral-type input-output representation. Notwithstanding this, using the Algorithm 2 is possible due to Lemma 25 since system (2) should be linearizable by additive output injections, as defined in Márquez-Martínez *et al.* (2002), to have a solution of Problems 2, and 3. Moreover, Lemma 25 is not a sufficient condition. Example 8 shows that Problem 1 and Problem 2 are not equivalent. Note that for the delay-free case the equivalence of this two problems is true.

Example 8 *The dynamical system defined by the delay-differential equation*

$$\begin{aligned} \dot{x}_1(t) &= x_1(t-1) + x_1(t-2) + x_2(t)x_2(t-2), \\ \dot{x}_2(t) &= 0, \\ y(t) &= x_1(t) + x_1(t-1), \end{aligned} \tag{115}$$

has an input-output equation

$$\ddot{y}(t) = \dot{y}(t-1) + \dot{y}(t-2), \tag{116}$$

that, through the change of variables $z_1(t) = y(t)$ *and* $z_2(t) = \dot{y}(t) - y(t-1) - y(t-2)$*, can be taken into the linear state-space representation*

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) + z_1(t-1) + z_1(t-2), \\ \dot{z}_2(t) &= 0, \\ y(t) &= z_1(t). \end{aligned} \tag{117}$$

Nevertheless, (115) *and* (117) *are not related by an invertible change of coordinates.* ◀

The above must be considered in the procedure for finding a solution of the problem of equivalence aimed in this chapter.

5.2.1 Effective Computation of the Input-Output Equation

The rest of the section is dedicated to present results, conditions to take the system into the input-output equation without a transformation of the state-space variable into the output variable. This allows to use algebraical computation strategies as the Euclidian division and concepts as the Smith form to find a solution of the addressed problem. This results were reported in García-Ramírez *et al.* (2016).

Lemma 28 *Assume that the given system is in its state-space representation, and let*

$\bar{A}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) \in \mathcal{K}^{(2n+1)\times 2n}(\delta]$ *be*

$$\bar{A}(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta) = \begin{pmatrix} \hat{A}(\mathbf{x}_{[s]},\bar{\mathbf{u}}_{[s]},\delta) & \hat{B}(\mathbf{x}_{[s]},\bar{\mathbf{u}}_{[s]},\delta) \\ 0 & \mathbb{I} \end{pmatrix} \tag{118}$$

where setting $\bar{\mathbf{u}}(t) = (\mathbf{u}^T(t), \dot{\mathbf{u}}^T(t), \cdots, (\mathbf{u}^{(\mathbf{n-1})})^T)$*,*

$$\hat{A}(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta) \;=\; \sum_{i=0}^{s} \tfrac{\partial(H^{(n)},H^{(n-1)},\ldots,H)}{\partial x(t-i)}\delta^i, \quad \hat{B}(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta) = \sum_{i=0}^{s} \tfrac{\partial(H^{(n)},H^{(n-1)},\ldots,H)}{\partial \bar{u}(t-i)}\delta^i. \tag{119}$$

Then, the given system admits an input-output equation of retarded type, if and only if the left-annihilator of the matrix $\bar{A}(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta)$ *is generated by a normalized covector* $\lambda(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta)$*, as defined in Definition 7.*

Proof. Consider the set of equation

$$0 = \lambda(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) \begin{pmatrix} dy^{(n)} \\ dy^{(n-1)} \\ \vdots \\ d\dot{y} \\ dy \\ du^{(n-1)} \\ \vdots \\ d\dot{u} \\ du \end{pmatrix} = \lambda(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)\bar{A}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta) \begin{pmatrix} \frac{dx}{d\bar{u}} \end{pmatrix} \tag{120}$$

Since the dimension of the columns of $\bar{A}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)$ is $2n+1$, and the system is claimed to be observable, $rank(\bar{A}(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)) = 2n$, so that there is one solution in the left kernel. If the system admits an input-output equation of retarded type, then there exists a $\lambda = [\chi_n, \cdots, \chi_0, \mu_{n-1}, \cdots, \mu_0]$ with $\chi_n = 1$ satisfying equation (120). Conversely if λ is a normalized vector, then $\chi_n = 1 \neq 0$

$$dy^{(n)} = \sum_{j=0}^{n-1} p_j(\bar{\mathbf{y}}, \bar{\mathbf{u}}, \delta) dy^{(j)} + \sum_{j=0}^{n-1} q_j(\bar{\mathbf{y}}, \bar{\mathbf{u}}, \delta) du^{(j)}$$

which, as before, ensures that the input–output equation is of retarded type. ∎

The system is claimed to be observable so $rank(A(x_{[s]}, u_{[s]}, \delta)) = n$. Then, the problem of finding the elements $\lambda_i, \mu_j,\ i = j = 0, \ldots n-1$ of $\lambda(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta)$ is the same problem of finding the kernel of the matrix $A(x_{[s]}, u_{[s]}, \delta)$. The next result is presented since it is useful in the effective computation of the vector $\lambda(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta)$.

Lemma 29 *Let $\bar{A}(x_{[s]}, u_{[s]}, \delta) \in \mathcal{K}^{(n+1)\times n}(\delta]$ be a matrix of row rank $2n-1$, $\hat{S}(x_{[s]}, u_{[s]}, \delta) \in \mathcal{K}^{(n+1)\times n}(\delta]$ its Smith pre-form such that the equality*

$$\bar{A}(x_{[s]}, u_{[s]}, \delta) = P(x_{[s]}, u_{[s]}, \delta)\hat{S}(x_{[s]}, u_{[s]}, \delta)Q(x_{[s]}, u_{[s]}, \delta), \tag{121}$$

is satisfied with $Q \in R^{2n\times 2n}$ a full rank matrix and $P(x_{[s]}, u_{[s]}, \delta) \in \mathcal{K}^{(2n+1)\times(2n+1)}(\delta]$ unimodular. Then, the $n-$ th row of $P^{-1}(x_{[s]}, u_{[s]}, \delta)$ is a left-annihilator of $\bar{A}(x_{[s]}, u_{[s]}, \delta)$.

Proof. $P(x_{[s]}, u_{[s]}, \delta)$ is a unimodular matrix (121) can be rewriting as

$$P^{-1}(x_{[s]}, u_{[s]}, \delta)\bar{A}(x_{[s]}, u_{[s]}, \delta) = \hat{S}(x_{[s]}, u_{[s]}, \delta)Q, \tag{122}$$

$\hat{S}(x_{[s]}, u_{[s]}, \delta)$ has the form

$$\hat{S}(x_{[s]}, u_{[s]}, \delta) = \begin{pmatrix} \alpha_{1,1}(x_{[s]}, \delta) & \alpha_{1,2}(x_{[s]}, \delta) & \dots & \alpha_{1,2n}(x_{[s]}, \delta) \\ 0 & \alpha_{2,2}(x_{[s]}, \delta) & \dots & \alpha_{2,2n}(x_{[s]}, \delta) \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & \alpha_{2n,2n}(x_{[s]}, \delta) \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

and Q is a column-shift transformation matrix that does not perform row operations. The above implies that the last row of $P^{-1}(x_{[s]}, u_{[s]}, \delta)$ annihilates the columns of $A(x_{[s]}, u_{[s]}, \delta)$ by the left since the last row of $\hat{S}(x_{[s]}, u_{[s]}, \delta)Q$ is zero. ■

Considering Lemma 29, it is possible to compute the vector $\lambda(\mathbf{x}_{[s]}, \bar{\mathbf{u}}_{[s]}, \delta)$ using the Smith pre-form algorithm, and assigning the entries of the last row of $P^{-1}(x_{[s]}, u_{[s]}, \delta)$ as the elements of the vector $\begin{pmatrix} 1 & \lambda_{n-1} & \cdots & \lambda_0 & \mu_{n-1} & \cdots & \mu_0 \end{pmatrix}$. See Garate-García (2006), or García-Ramírez (2011) to find implemented results about this calculation.

Example 9 *Consider the dynamical retarded system given by the equation*

$$\begin{aligned}
\dot{x}_1(t) = {} & 2\big(a_0x_2^2(t-2) - a_0x_1\,(t-1) + (x_1\,(t-5) - x_2^2(t-6) - a_1)\,x_1\,(t-4) + \\
& (x_2^2(t-6) - x_1\,(t-5) + a_1)\,x_2^2(t-5)\big)x_2(t-1) + \\
& x_2(t) + 2x_2\,(t-1)\,\mathrm{u}\,(t-1)\,, \\
\dot{x}_2(t) = {} & (x_1\,(t-4) - x_2^2(t-5) - a_1)\,x_1\,(t-3) - a_0x_1\,(t) + a_0x_2^2(t-1) + \\
& (-x_1(t-4) + x_2^2(t-5) + a_1)\,x_2^2(t-4) + \mathrm{u}\,(t)\,, \\
y(t) \;\; = {} & \;\; x_1(t-3) - x_2^2(t-4).
\end{aligned} \tag{123}$$

The first n-derivatives with respect to time of the output are

$$\begin{aligned}
y(t) &= x_1(t-3) - x_2^2(t-4), \\
\dot{y}(t) &= x_2\,(t-3)\,, \\
\ddot{y}(t) &= -a_0x_1\,(t-3) + a_0x_2^2(t-4) + (x_1\,(t-7) - x_2^2(t-8) - a_1)\,x_1\,(t-6) + \\
& \quad (-x_1\,(t-7) + x_2^2(t-8) + a_1)\,x_2^2(t-7) + \mathrm{u}\,(t-3)\,.
\end{aligned}$$

From the differential form of the input-output equation represented as

$$d\psi = \alpha\lambda \begin{pmatrix}
\begin{array}{c}\left(x_1\,(t-6) - x_2^2(t-7)\right)\delta^7 + \\ \left(x_1\,(t-7) - x_2^2(t-8) - a_1\right)\delta^6 - \\ a_0\delta^3\end{array} & \begin{array}{c}\left(2x_2\,(t-8)\,x_2^2(t-7) - 2x_2\,(t-8)\,x_1\,(t-6)\right)\delta^8 + \\ \left(2x_2^2(t-8) - 2x_1\,(t-7) + 2a_1\right)x_2\,(t-7)\,\delta^7 + \\ 2a_0x_2\,(t-4)\,\delta^4\end{array} & 0 & \delta^3 \\
0 & \delta^3 & 0 & 0 \\
\delta^3 & -2x_2\,(t-4)\,\delta^4 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix} dx_1(t) \\ dx_2(t) \\ d\dot{u}(t) \\ du(t) \end{pmatrix}$$

From the Smith preform calculation the matrices that correspond to the equality (122) *are*

$$P^{-1}(x_{[s]}, u_{[s]}, \delta) = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\left(-x_1\,(t-6) + x_2^2\,(t-7)\right)\,\delta^4 + \left(-x_1\,(t-7) + x_2^2\,(t-8) + a1\right)\,\delta^3 + a0 & 0 & 1 & -\delta^3 & 0
\end{pmatrix}$$

and

$$\hat{S}(x_{[s]}, u_{[s]}, \delta)Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \delta^3 & -2\,x_2\,(t-4)\;\delta^4 & 0 & 0 \\ 0 & \delta^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The vector $\lambda(\mathbf{x}_{[s]}, \delta)$ *is then defined as*

$$\lambda(\mathbf{x}_{[s]}, \delta) = \begin{pmatrix} 1 & 0 & (-x_1\,(t-6) + x_2^2(t-7))\,\delta^4 + (-x_1\,(t-7) + x_2^2(t-8) + a_1)\,\delta^3 + a_0 & 0 & -\delta^3 \end{pmatrix}.$$

◀

5.3 Observer Design Through Linearization, Up To Input-Output Injection, Via Invertible Transformation.

In this section a constructive effective computational solution of Problem 3 is presented. This result were reported in García-Ramírez *et al.* (2016). The procedure to be followed starts finding the normalized vector $\lambda(\mathbf{x}, \bar{\mathbf{u}}, \delta)$ that defines the coefficients, in the state-space variables, of the differential form of the input-output equation of the system. Then, if the system is equivalent to the form (104), using $\lambda(\mathbf{x}, \bar{\mathbf{u}}, \delta)$ the computation of the injection functions $\Phi_i(\mathbf{x}, \mathbf{u}, \delta)$, $i = 1, \ldots n$ can be used to calculate

$$\begin{aligned} h_1(t) &= y(t) \\ h_2(t) &= \dot{y}(t) - \Phi_1(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}) \\ &\vdots \\ h_{n-1}(t) &= y^{n-1}(t) - \sum_{i=1}^{n-1} \Phi_i^{(n-i-1)}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]}), \end{aligned} \tag{124}$$

that, if the Problem 2 is solvable for the system, eliminates the non linearities from the first $n-1$ time-derivatives of the input, allowing to compute the desired change of coordinates. Note that, if the linear part of the linearized system is delay-free, which means that is equivalent to equation (109), $\mathbf{z}(t) = \mathbf{h}(t)$ defines an invertible change of coordinates. The Algorithm 3 computes, if possible, the injection functions $\Phi_i(\mathbf{x}, \mathbf{u}, \delta)$, $i = 1, \ldots n$, and $\mathbf{h}(t)$

from the normalized vector $\lambda(\mathbf{x}, \bar{\mathbf{u}}, \delta)$.

Algorithm 3

Let $\lambda(\mathbf{x}, \bar{\mathbf{u}}, \delta) := [1, \chi^0_{n-1}, \cdots, \chi^0_0, \mu^0_{n-1}, \cdots, \mu^0_0]$ *be a normalized covector satisfying Lemma 28.*

Set

$$\Psi_1 := -\sum_{i=0}^{n-1} \chi^0_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) dy^{(i)}(t) - \sum_{i=0}^{n-1} \mu^0_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) du^{(i)}(t). \tag{125}$$

and set

$$dh_0 := dH(\mathbf{x}_{[s]}) \tag{126}$$

STEP 1. *Set* $\omega_1 := -\chi^0_{n-1}(\mathbf{x}, \bar{\mathbf{u}}, \delta) dy(t) - \mu^0_{n-1}(\mathbf{x}, \bar{\mathbf{u}}, \delta) du(t)$

Check: $d\omega_1 = 0$*?*

NO: Stop, YES: Compute $\Phi_1(\mathbf{x}, \mathbf{u}, \delta)$ *such that* $\omega_1 = d\Phi_1(\mathbf{x}, \mathbf{u}, \delta)$*, and set*

$$dh_1(x) := d\dot{H}(x(t)) - d\Phi_1(\mathbf{x}, \mathbf{u}, \delta) \tag{127}$$

and

$$\Psi_2 \quad := \quad -\textstyle\sum_{i=0}^{n-2} \chi^1_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) dy^{(i)}(t) - \sum_{i=0}^{n-2} \mu^1_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) du^{(i)}(t). \tag{128}$$

with

$$\begin{aligned} \chi^1_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) &= \chi^0_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) - \binom{n-1}{n-1-i} (\chi^0_{n-1}(\mathbf{x}, \bar{\mathbf{u}}, \delta))^{(n-1-i)} \\ \mu^1_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) &= \mu^0_i(\mathbf{x}, \bar{\mathbf{u}}, \delta) - \binom{n-1}{n-1-i} (\mu^0_{n-1}(\mathbf{x}, \bar{\mathbf{u}}, \delta))^{(n-1-i)} \end{aligned}$$

STEP k. *Set* $\omega_k := -\chi^{k-1}_{n-k}(\mathbf{x}, \bar{\mathbf{u}}, \delta) dy(t) - \mu^{k-1}_{n-k}(\mathbf{x}, \bar{\mathbf{u}}, \delta) du(t)$

Check: $d\omega_k = 0$*?*

NO: Stop, YES: Compute $\Phi_k(\mathbf{x},\mathbf{u},\delta)$ *such that* $\omega_k = d\Phi_k(\mathbf{x},\mathbf{u},\delta)$. *Set*

$$dh_k(x) := dH(x(t))^{(k)} - \sum_{j=0}^{k-1} d\Phi_{k-j}(\mathbf{x},\bar{\mathbf{u}},\delta)^{(j)}, \tag{129}$$

and

$$\Psi_{k+1} \quad := \quad -\textstyle\sum_{i=0}^{n-k-1} \chi_i^k(\mathbf{x},\bar{\mathbf{u}},\delta)dy^{(i)}(t) - \sum_{i=0}^{n-k-1} \mu_i^k(\mathbf{x},\bar{\mathbf{u}},\delta)du^{(i)}(t), \tag{130}$$

with

$$\begin{aligned}
\chi_i^k(\mathbf{x},\bar{\mathbf{u}},\delta) &= \chi_i^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta) - \binom{n-k}{n-k-i}(\chi_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)} \\
\mu_i^k(\mathbf{x},\bar{\mathbf{u}},\delta) &= \mu_i^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta) - \binom{n-k}{n-k-i}(\mu_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)}
\end{aligned}$$

♦

The following proposition ensures that computations of Algorithm 3, that are made on the state-space variables, define the input-output injection functions $\Phi_i(y(t),\cdots,y(t-s),u(t),\cdots,u(t-s))$ from the input-output representation (104).

Proposition 30 *Assume that* ω_i *in Algorithm 3 is an exact differential for* $i = 1,\cdots k$. *Then*

- *i)* $\omega_i = d\Phi_i(y(t),\cdots,y(t-s),u(t),\cdots,u(t-s))$
- *ii)* $\Psi_i = y^{(n)}(t) - \sum_{l=1}^{i-1} \Phi_l^{(n-l)}(\bar{\mathbf{y}},\bar{\mathbf{u}})$

Proof. By construction

$$\omega_i = -\chi_{n-i}^{i-1}(x,\bar{u},\delta)dy - \mu_{n-i}^{i-1}(x,\bar{u},\delta)du.$$

Since ω_i is an exact differential, then necessarily it is only a function of $y(t), u(t)$ and their delays, which proves i).

As for ii), the proof is iterative. Ψ_1 is computed starting from the normalized covector λ and thus $\Psi_1 = y^{(n)}(t)$. Assume that ii) is true from k, then

$$\omega_k = -\chi_{n-k}^{k-1}(x,\bar{u},\delta)dy - \mu_{n-k}^{k-1}(x,\bar{u},\delta)du = d\Phi_k(y,u)$$

Accordingly

$$d\Phi_k^{(n-k)} = -\sum_{\ell=0}^{n-k}\binom{n-k}{\ell}\left[\left(\chi_{n-k}^{k-1}(x,\bar{u},\delta)\right)^{(\ell)}dy^{(n-k-\ell)} - \left(\mu_{n-k}^{k-1}(x,\bar{u},\delta)\right)^{(\ell)}du^{(n-k-\ell)}\right].$$

It follows that

$$\begin{aligned}
\Psi_{k+1} &= -\sum_{i=0}^{n-k-1}\chi_i^k(\mathbf{x},\bar{\mathbf{u}},\delta)dy^{(i)}(t) - \sum_{i=0}^{n-k-1}\mu_i^k(\mathbf{x},\bar{\mathbf{u}},\delta)du^{(i)}(t)\\
&= -\sum_{i=0}^{n-k-1}\left(\chi_i^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta) - \binom{n-k}{n-k-i}(\chi_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)}\right)dy^{(i)}\\
&\quad -\sum_{i=0}^{n-k-1}\left(\mu_i^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta) - \binom{n-k}{n-k-i}(\mu_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)}\right)du^{(i)}\\
&= \Psi_k + \sum_{i=0}^{n-k}\left(\binom{n-k}{n-k-i}(\chi_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)}dy^{(i)} + \right.\\
&\quad \left.\binom{n-k}{n-k-i}(\mu_{n-k}^{k-1}(\mathbf{x},\bar{\mathbf{u}},\delta))^{(n-k-i)}du^{(i)}\right)\\
&= \Psi_k - d\Phi_k^{(n-k)} = y^{(n)}(t) - \sum_{j=1}^{k}d\Phi_j^{(n-j)}
\end{aligned}$$

■

Now, the main contribution from García-Ramírez *et al.* (2016), that solves Problem 2 is presented in Theorem 31.

Theorem 31 *Problem 2 is solvable if and only if*

i) the system admits an input–output equation of retarded type

ii) The one-forms ω_i defined by Algorithm 3 are exact for all $i = 1,\ldots,n$.

iii) There exists a polymodular matrix $T(\mathbf{x}_{[p,j]},\delta)$ and a full-rank matrix $Q(\delta)\in\mathbb{R}[\delta]$ such that $Q(\delta)T(\mathbf{x}_{[p,j]},\delta)dx(t+p) = P(\mathbf{x}_{[s]},\delta)dx(t) = (dh_0^T,\ldots,dh_{n-1}^T)^T$ from Algorithm 3.

Proof. From Lemma 25, it follows that system (2) is linearizable by additive input-output injections only if $i)$ stands. Assume now that the system is already in the form (95), and apply Algorithm 3. Because of its structure, the differential of its input-output equation is given by (104), that is,

$$dy^{(n)} = -\sum_{i=0}^{n-1}\sigma_i dy^{(i)}(t) + \sum_{k=1}^{n}\sum_{i=0}^{k-1}\sigma_k C(\delta)A^i(\delta)d\varphi(\mathbf{y}_{[s]},\mathbf{u}_{[s]})^{(k-i-1)},$$

Accordingly one gets that, starting from $dh_0 = C(\delta)dz$, at the first step

$$\begin{aligned}\omega_1 &= -\sigma_{n-1}dy + C(\delta)d\varphi = d\Phi_1\\ dh_1 &= C(\delta)A(\delta)dz + \sigma_{n-1}dy\end{aligned}$$

and at step k

$$\begin{aligned}\omega_k &= -\sigma_{n-k}dy + \sum_{j=0}^{k-1}\sigma_{n-k+1+j}C(\delta)A^j(\delta)d\varphi = d\Phi_k\\ dh_k &= \sum_{j=0}^{k}\sigma_{n-j}C(\delta)A(\delta)^j dz(t)\end{aligned}$$

which proves that the ω_i's must be exact one-forms. Furthermore, in the x-coordinates one thus gets

$$d\hat{\mathbf{h}} = \begin{pmatrix}1 & 0 & 0 & \dots & 0\\ \sigma_{n-1} & 1 & 0 & \dots & 0\\ \sigma_{n-2} & \sigma_{n-1} & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \sigma_1 & \sigma_2 & \sigma_3 & \dots & 1\end{pmatrix}\begin{pmatrix}C(\delta)\\ C(\delta)A(\delta)\\ C(\delta)A(\delta)^2\\ \vdots\\ C(\delta)A(\delta)^{n-1}\end{pmatrix}dz(t) = Q(\delta)dz(t). \qquad (131)$$

Since by assumption $z(t) = \phi(x(t+p),\cdots,x(t-j))$, $dz(t) = T(\mathbf{x}_{[p,j]},\delta)dx(t+p)$, we have that

$$d\hat{\mathbf{h}} = Q(\delta)T(\mathbf{x}_{[p,j]},\delta)dx(t+p) = P(\mathbf{x}_{[0,s]},\delta)dx(t)$$

which proves the necessity of iii).

For the sufficiency, according to iii) there exists $z(t)=\phi(x(t+p),\cdots,x(t-j))$, such that $dz(t)=T(\mathbf{x}_{[p,j]},\delta)dx(t+p)$. Since conditions $i)$ and $ii)$ are verified, in the z–coordinates the output of the Algorithm 3 is given by

$$\begin{pmatrix} dy \\ d\dot{y}-d\varphi_1(y,u) \\ dy^{(2)}-d\dot{\varphi}_1(y,u)-d\varphi_2(y,u) \\ \vdots \\ dy^{(n-1)}-d\varphi_1^{(n-2)}(y,u)-\ldots-d\varphi_{n-1}(y,u) \end{pmatrix} = Q(\delta)dz. \tag{132}$$

Differentiating equation (132) and denoting by $q_i(\delta)$ the i-th row of the matrix $Q(\delta)$

$$\begin{aligned} Q(\delta)d\dot{z} &= \begin{pmatrix} d\dot{y} \\ d\ddot{y}-d\dot{\varphi}_1(y,u) \\ dy^{(3)}-d\ddot{\varphi}_1(y,u)-d\dot{\varphi}_2(y,u) \\ \vdots \\ dy^{(n)}-d\varphi_1^{(n-1)}(y,u)-\ldots-d\dot{\varphi}_{n-1}(y,u) \end{pmatrix} = \begin{pmatrix} q_2(\delta)dz+d\varphi_1 \\ \vdots \\ q_n(\delta)dz+d\varphi_{n-1} \\ d\varphi_n \end{pmatrix} \\ &= \begin{pmatrix} q_2(\delta) \\ \vdots \\ q_n(\delta) \\ 0 \end{pmatrix} dz + d\varphi = \bar{A}(\delta)dz+d\varphi \end{aligned} \tag{133}$$

Multiplying by the adjunct matrix $Q^{(a)}(\delta)$ we get

$$\begin{pmatrix} \bar{q}_1(\delta) & & \\ & \ddots & \\ & & \bar{q}_n(\delta) \end{pmatrix} d\dot{z} = \hat{A}(\delta)dz + d\hat{\varphi}$$

Using the identity of polynomials one thus gets that

$$d\dot{z} = A(\delta)dz + d\Psi(y,u)$$

which ends the proof. ■

Theorem 31 solves 2 allowing to take the system in the canonical linear, up to input-output injection, form (95) for which a Luenberger-type observer described by equation (96) can be design. Nevertheless, an invertible change of coordinates of $\boldsymbol{\zeta}(t) = \phi(\boldsymbol{\xi}_{[p,s]})$ can lead to a non causal observer dynamics. To deal with this consider $\boldsymbol{\xi}(t) = \bar{\phi}(\boldsymbol{\zeta}_{[p,s]})$ that fulfills $\bar{\phi}(\boldsymbol{\zeta}_{[p',s']}) \mid_{\mathbf{z}(t)=\phi(\boldsymbol{\xi}_{[p,s]})} = \mathbf{x}(t)$, and define a change of variable is proposed as follows

$$\bar{\boldsymbol{\xi}}(t) = \begin{pmatrix} \xi_1(t-p_1) \\ \vdots \\ \xi_n(t-p_n) \end{pmatrix} = \begin{pmatrix} \bar{\phi}_1(\boldsymbol{\zeta}_{[\mathbf{0},\mathbf{s}'+\mathbf{p_1}]}) \\ \vdots \\ \bar{\phi}_n(\boldsymbol{\zeta}_{[\mathbf{0},\mathbf{s}'+\mathbf{p_n}]}) \end{pmatrix}$$

that takes the observer into a non-anticipative dynamical system.
The following example illustrates how the results given in this section can be used for the observation of a dynamical system.

Example 10 *Consider the equation describing system* (123) *from the Example 9. The normalized vector $\lambda(\mathbf{x}_{[s]}, \delta)$ is then defined as*

$$\lambda(\mathbf{x}_{[s]}, \delta) = \begin{pmatrix} 1 & 0 & \left(-x_1(t-6) + x_2^2(t-7)\right)\delta^4 + \left(-x_1(t-7) + x_2^2(t-8) + a_1\right)\delta^3 + a_0 & 0 & -\delta^3 \end{pmatrix}.$$

is used as an input for the Algorithm 3. First set

$$\begin{aligned} \Psi_1 &:= -\Big(\left(-x_1(t-6) + x_2^2(t-7)\right)\delta^4 + \left(-x_1(t-7) + x_2^2(t-8) + a_1\right)\delta^3 + a_0\Big)dy(t) + \\ &\quad \delta^3 du(t) \\ dh_0 &= \delta^3 dx_1(t) - 2x_2(t-4)\delta^4 dx_2(t) \end{aligned}$$

STEP 1. *Since $\omega_1 = 0$ is exact, and it is possible to set*

$$\begin{aligned} \Psi_2 &:= -\Big(\left(-x_1(t-6) + x_2^2(t-7)\right)\delta^4 + \left(-x_1(t-7) + x_2^2(t-8) + a_1\right)\delta^3 + a_0\Big)dy(t) + \\ &\quad \delta^3 du(t), \\ dh_1 &= \delta^3 dx_2(t). \end{aligned}$$

Since $\omega_2 = \Psi_2 = d\Big(a_0 y(t) + a_1 y(t-3) + y(t-3)y(t-4)\Big)$ is exact, the algorithm ends.

These results allow to find an invertible change of coordinates defined from the differential form given by

$$\begin{array}{rclcl} dh_0 & = & \delta^3 dx_1(t) - 2x_2(t-4)\delta^4 dx_2(t) & = & d\left(x_1(t-3) - x_2^2(t-4)\right), \\ dh_1 & = & \delta^3 dx_2(t) & = & d\left(x_2(t-3)\right), \end{array} \tag{134}$$

with an inverse described by the equations

$$\begin{array}{rcl} x_1(t-3) & = & z_1(t) + z_2^2(t-1), \\ x_2(t-3) & = & z_2(t) \end{array}$$

that takes system (123) *into the form*

$$\begin{array}{rcl} \dot{z}_1(t) & = & z_2\left(t\right), \\ \dot{z}_2(t) & = & -a_0 z_1\left(t\right) - a_1 z_1\left(t-3\right) + z_1\left(t-3\right) z_1\left(t-4\right) + u(t-3), \\ y(t) & = & z_1(t), \end{array} \tag{135}$$

which is in the desired form. Note that the use of this non bicausal change of coordinates allows to compute the delayed state only. A Luenberger-type observer for system (135) *is proposed as*

$$\begin{array}{rcl} \dot{\zeta}_1(t) & = & \zeta_2(t) + k_1\big(\zeta_1(t) - z_1(t)\big), \\ \dot{\zeta}_2(t) & = & k_2\big(\zeta_1(t) - z_1(t)\big) - a_0 y\left(t\right) - a_1 y\left(t-3\right) + y\left(t-3\right) y\left(t-4\right) + u(t-3). \end{array} \tag{136}$$

That defines an error dynamics defined as

$$\begin{array}{rcl} \dot{\epsilon}_1(t) & = & \epsilon_2(t) + k_1 \epsilon_1(t), \\ \dot{\epsilon}_2(t) & = & k_2 \epsilon_1(t). \end{array} \tag{137}$$

In the coordinates $\bar{\boldsymbol{\xi}}(t)$

$$\begin{array}{rcccl} \bar{\xi}_1(t) & = & \xi_1(t-3) & = & \zeta_1(t) + \zeta_2^2(t-1), \\ \bar{\xi}_2(t) & = & \xi_2(t-3) & = & \zeta_2(t), \end{array}$$

the observed is defined

$$
\begin{aligned}
\dot{\bar{\xi}}_1(t) &= \bar{\xi}_2(t) + 2\bar{\xi}_2(t-1)^2\big(y(t-4)y(t-5) - a_0 y(t-1) - a_1 y(t-4) + \\
&\quad k_2\big(\bar{\xi}_1(t-4) - \bar{\xi}_2^2(t-5) - x_1(t-4) + x_2^2(t-5)\big) + u(t-4)\big) + \\
&\quad k_1\big(\bar{\xi}_1(t-3) - \bar{\xi}_2^2(t-4) - x_1(t-3) + x_2^2(t-4)\big), \\
\dot{\bar{\xi}}_2(t) &= k_2\big(\bar{\xi}_1(t-3) - \bar{\xi}_2^2(t-4) - x_1(t-3) + x_2^2(t-4)\big) - a_0 y\,(t) - \\
&\quad a_1 y\,(t-3) + y\,(t-3)\,y\,(t-4) + u(t-3).
\end{aligned}
\tag{138}
$$

Since the equivalent linear system has a state feedback linearizable canonical form, a linearization input is chosen as to drive the system into the time delay linear differential equation

$$
\begin{aligned}
\dot{z}_1(t) &= z_2(t), \\
\dot{z}_2(t) &= -a_0 z_1(t) - a_1 z_1(t-3) - \bar{k}\zeta_1(t-3).
\end{aligned}
\tag{139}
$$

Such an input in the original coordinates is $u(t) = -(-a_1 + \bar{k})\zeta_1(t-3) - \big(\xi_1(t) + \xi_2^2(t-3)\big)\big(\xi_1(t-1) + \xi_2^2(t-4)\big)$*. Note that* $-(-a_1 + \bar{k})$ *should be taken considering the stability region as defined in Theorem 1 or, for a delay-free approximation, the stability region defined in Proposition 11 or 13.*

In Figure 22 numerical simulations of the state variable $\mathbf{x}(t)$*, and the observed delayed state* $\mathbf{x}(t-3)$*. For this simulation the base time-delay of the system is taken as* $1/3$ *units of time, the system parameter values are* $a_0 = 5$*, and* $a_1 = -3$*, and the observer gain values are set as* $k_1 = -1.414$*, and* $k_2 = -1$*.*

The error of the observed signals are compared shifting the solution of the system as is presented in Figure 23. ◀

5.4 Observer Design Through Linearization, Up To Input-Output Injection, Via Bi-causal Transformation.

In this section, Problem 3 is discussed. Problem 3 consists in the search of a canonical structure with delays in the linear part, and a nonlinear part depending on the input, the output, and their respective delays only. In Section 5.3 it was shown that a non bicausal invertible change of coordinates allows to design an observer to compute the retarded

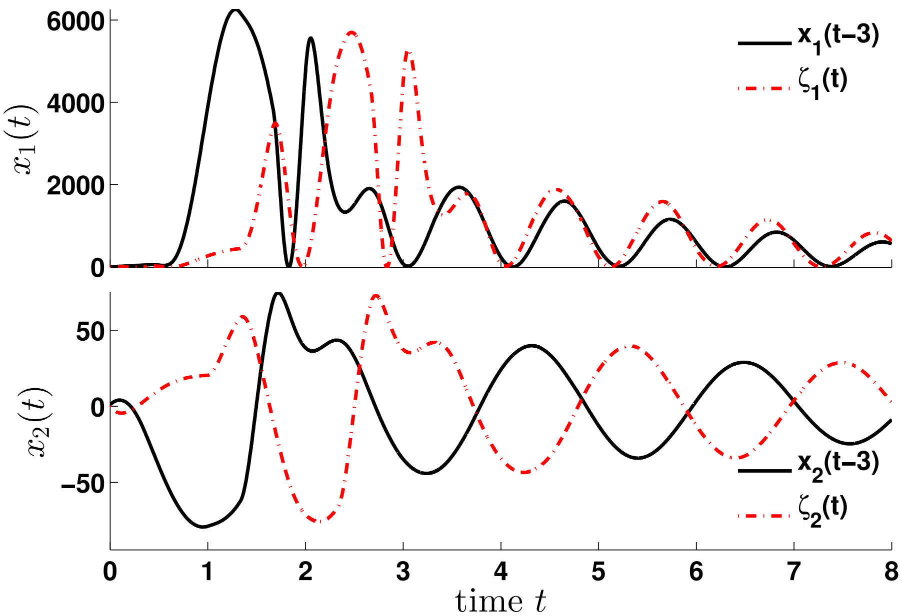


Figure 22: Time-delayed observation of the NLDS (123)**.**

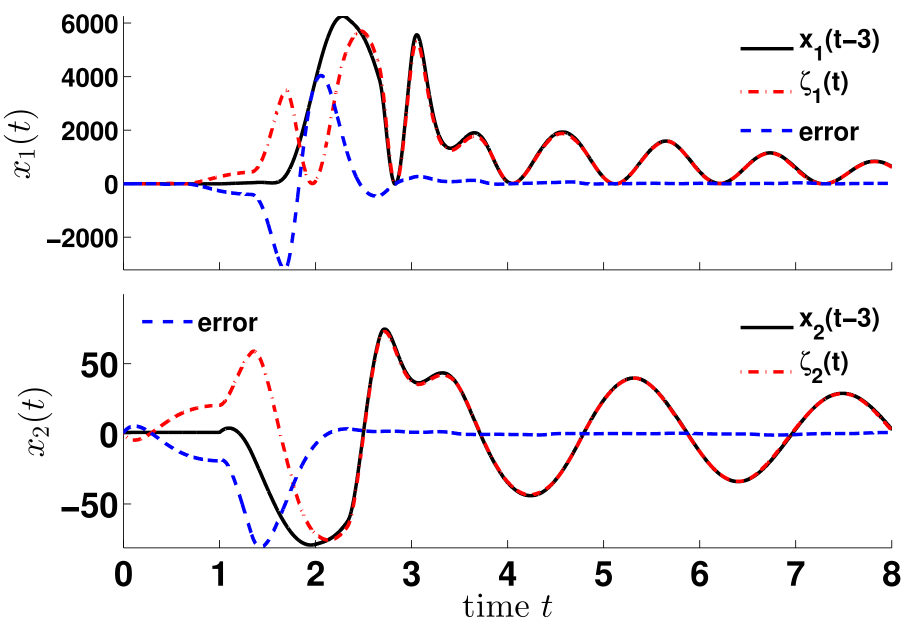


Figure 23: Error of the time-delayed observation of the NLDS (123)**.**

state variable of the system. Nevertheless, a bicausal change of coordinates permit to calculate the actual value of the state variables, for example by means of a Luenberger-type observer, since the observer dynamics is not affected by variables that depends of the future. Note also that a bicausal transformation allows to keep invariant the strong and weak observability properties of the dynamical system. As it is pointed out in García-Ramírez *et al.* (2016), non bicausal change of coordinates can modify these properties. Consider, for instance, next example:

Example 11 *The strong observable system*

$$\begin{aligned}\dot{x}(t) &= ax(t),\\ y(t) &= x(t).\end{aligned}$$

The change of coordinates $z = x(t+1)$ *takes the system into the form*

$$\begin{aligned}\dot{z}(t) &= az(t),\\ y(t) &= z(t-1),\end{aligned} \tag{140}$$

which is weakly-observable.

Now, considering Theorem 31 to design an observer for (140) *the change of coordinates is computed*

$$\begin{aligned}y(t) &= h_0 = \bar{z}(t) = z(t-1),\\ \dot{y}(t) &= \varphi = az(t-1).\end{aligned}$$

The equations of the system in the new coordinates are

$$\begin{aligned}\dot{\bar{z}}(t) &= a\bar{z}(t-1),\\ y(t) &= \bar{z}(t).\end{aligned} \tag{141}$$

an observer defined by the equation

$$\dot{\bar{\zeta}}(t) = k\big(\bar{\zeta}(t) - \bar{z}(t)\big) - a\bar{z}(t-1),$$

with an error equation $\bar{e}(t) = \bar{z}(t) - \bar{\zeta}(t)$ *defines an error dynamical system*

$$\dot{\bar{e}}(t) = k\bar{e}(t). \tag{142}$$

Since the change of coordinates is not bicausal any observer designed using system (141) *allow to compute the delayed state* $z(t-1)$*. System* (140) *is already in the form* (95) *so it* *is possible to design an observer*

$$\dot{\zeta}(t) = a\zeta + k\big(\zeta(t-1) - z(t-1)\big),$$

with an error $e(t) = z(t) - \zeta(t)$ *with dynamic*

$$\dot{e}(t) = ae(t) + ke(t-1). \tag{143}$$

This means that for system (140)*, the design of an observer whose solution converges to the present state value is feasible. Note also that error dynamics* (142) *is linear delay-free, and* (143) *is a linear time-delay system, so time-delay stability considerations should be taken for* (143) *while for* (142) *classical delay-free techniques may be used.* ◀

To solve Problem 3, the next corollary of 31 will be used.

Corollary 32 *Problem 2 is solvable only if*

i) the system admits an input–output equation of retarded type

ii) The one-forms ω_i *defined by Algorithm 3 are exact for all* $i = 1, \ldots, n$*.*

iii) There exists a bicausal matrix $T(\mathbf{x}_{[p,j]}, \delta)$ *and a full rank matrix* $Q(\delta) \in \mathbb{R}[\delta]$ *such that* $Q(\delta)T(\mathbf{x}_{[p]}, \delta)dx(t) = P(\mathbf{x}_{[s]}, \delta)dx(t) = (dh_0^T, \ldots, dh_{n-1}^T)^T$ *from Algorithm 3.*

Sufficiency of conditions on Corollary 32 are not sufficient, as it can be illustrated by the following example

Example 12 *Consider the system*

$$\begin{aligned} \dot{x}(t) &= x(t)u(t-1), \\ y(t) &= x(t-1). \end{aligned} \tag{144}$$

The system defined by equation (144) *is linear up to injection of the state variable that defines the output in the new coordinates, and fulfills the procedure described until now* *but is not in the form* (95)*, since the nonlinear terms depend on the future values of the output (e.g.* $x(t)u(t-1) = y(t+1)u(t-1)$*). This example shows that Problem 2 has a solution in the case of the input-output representation derived from* (144)*, but the existence of such a solution does not imply that Problem 3 is solvable because a bicausal change of coordinates to a system in the canonical form* (95) *does not exist.* ◁

In the next paragraphs a solution for Problem 3 is presented dealing with the dependence of noncausal output dependent non linearities as the ones presented by Example 12.

Without loss of generality, the differential derivative of equation (95) can be expressed as

$$\begin{aligned} d\dot{z}(t) &= A(\delta)dz(t) + \boldsymbol{\vartheta}(\delta)dy(t) + d\left(\hat{\varphi}(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})\right), \\ dy(t) &= C(\delta)dz(t), \end{aligned} \tag{145}$$

with $\boldsymbol{\vartheta}(\delta) \in \mathbb{R}^n[\delta]$. By means of the bicausal transformation $\mathbf{z}(t) = \varphi(\mathbf{x}_{[s]})$, defined through Corollary 32, the system in the new coordinates can be described, without loss of generality, by

$$\begin{aligned} \dot{z}_i &= \bar{f}_i(\mathbf{z}_{[s]}, \mathbf{u}_{[s]}), \qquad i = 1, \dots n \\ y(t) &= c_1(\delta)z_1(t). \end{aligned} \tag{146}$$

Using the notation established in (146), the linear part of the equation (145) is written as

$$d\left(a_i(\delta)\mathbf{z}(t)\right) = \sum_{j=1}^{n} a_{ij}(\delta)dz_j(t) + \vartheta_i(\delta)dy(t), \ i = 1, \dots n, \tag{147}$$

where $a_i(\delta)$ is defined by the *i-th* row of the matrix $A(\delta)$, and each $a_{ij}(\delta), \ i = 1, \dots n$ by its entries. The remaining part of the computation of

$$d\left(a_i(\delta)\mathbf{z}(t)\right) \wedge dy(t), \dots, \wedge dy(t-s) = \left(\sum_{j=1}^{n} a_{ij}(\delta)dz_j(t)\right) \wedge dy(t), \dots, \wedge dy(t-s) \tag{148}$$

is not generated by $span_{\mathbb{R}[\delta]}\{dy(t),\ldots,dy(t-s)\}$. Besides this, the part of the equation (145) that is generated by $dy(t)$ can be computed as

$$d\phi_i(\mathbf{y}_{[s]},\mathbf{u}_{[s]}) = d\dot{z}_i(t) - \sum_{j=1}^{n} a_{ij}(\delta)dz_j(t),\ i=1,\ldots n. \tag{149}$$

The following result can be stated.

Proposition 33 *Problem 3 is solvable if and only if*

(i) $d\omega_i = 0$ *for all* $\omega_i,\ i=0,\ldots,n$ *from Algorithm 3,*

(ii) There exists a unimodular transformation matrix $T(\mathbf{x}_{[s]},\delta) \in \mathcal{K}(\delta]$*, and a full-rank matrix* $Q(\delta) \in \mathbb{R}[\delta]$*, such that* $Q(\delta)T(\mathbf{x}_{[s]},\delta)dx = P(\mathbf{x}_{[s]},\delta)dx = (dh_0,\ldots,dh_{n-1})^T$ *from Algorithm 3,*

(iii) There exists $d\bar{h}_i = \sum_{j=1}^{n}\alpha_{ij}(\cdot,\delta)dz_j(t)$ *with* $\alpha_{ij} \in \mathbb{R}[\delta]\ \forall i=1,\ldots,n$ *such that* $d\bar{h}_i \wedge dy(t),\ldots,\wedge dy(t-s) = d\bar{f}_i(\mathbf{z}_{[s]},0)\wedge dy(t),\ldots,\wedge dy(t-s),\ i=1,\ldots n$*, and*

(iv) $d\bar{\phi}_i(\cdot,\cdot) \in span_{\mathcal{K}(\delta]}\{d\mathbf{y},\ldots,d\mathbf{y}^{(n-1)},d\mathbf{u},\ldots,d\mathbf{u}^{(n-1)}\}$*, with* $d\bar{\phi}_i(\cdot,\cdot) := d\dot{z}_i - d\bar{h}_i,\ i = 1,\ldots,n$*.*

Proof. Sufficiency: Because Problem 3 has a solution only if conditions (i) and (ii) are fulfilled, then a bicausal transformation $dz(t) = T(\mathbf{x}_{[s]},\delta)dx(t)$ must exist. If (iii) is fulfilled, then $d\bar{h}_i = \sum_{j=1}^{n}\alpha_{ij}(\cdot,\delta)dz_j(t) = \sum_{j=1}^{n}a_{ij}(\delta)dz_j(t),\ i=1,\ldots n$ as defined in (147), and $d\bar{\phi}_i(\cdot,\cdot) := d\dot{z}_i - d\bar{h}_i = d\phi_i(\mathbf{y}_{[s]},\mathbf{u}_{[s]}),\ i=1,\ldots n$ as defined in (149). Then, it is sufficient that the system in the new coordinates fulfills (iii) and (iv) to have an equivalent state-space representation in the form (95).

Necessity: The necessity of (iii) comes directly from the computation

$$\begin{aligned} d\bar{f}_i \wedge dy(t)\wedge\ldots\wedge dy(t-s) &= \left(\sum_{j=1}^{n}\sum_{k=0}^{s} a_{ij}^k dz_j(t-k)\right)\wedge dy(t)\wedge\ldots\wedge dy(t-s)+ \\ &\quad \left(d\left(\varphi(\mathbf{y}_{[s]},0)\right)\right)\wedge dy(t)\wedge\ldots\wedge dy(t-s) \\ &= \left(\sum_{j=1}^{n}\sum_{k=0}^{s} a_{ij}^k dz_i(t-k)\right)\wedge dy(t)\wedge\ldots\wedge dy(t-s) \\ &= d\bar{h}_i \wedge dy(t)\wedge\ldots\wedge dy(t-s) \end{aligned} \tag{150}$$

on (95). Let us assume now that the system (2) is linearizable by additive input-output injections but (iv) does not stand, which means that the nonlinear part of the system in the new coordinates is not a function of the output and the input only. From the equation (150) on (95) we get $d\bar{h} = A(\delta)dz$ and then it is possible to compute

$$d\left(\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})\right) = d\dot{z}(t) - A(\delta)dz(t). \tag{151}$$

but, because of the necessity of (i), (ii), and (iii), after the bicausal change of coordinates given by $dz(t) = T(\mathbf{x}_{[s]}, \delta)$, the system must be in the form (95), and from (151) $d\left(\varphi(\mathbf{y}_{[s]}, \mathbf{u}_{[s]})\right) \in span_{\mathcal{K}(\delta]}\{d\mathbf{y}, \ldots, d\mathbf{y}^{(n-1)}, d\mathbf{u}, \ldots, d\mathbf{u}^{(n-1)}\}$, which is a contradiction with the asseveration that (iv) does not hold. ■

The next example shows how, by means of Proposition 33, a bicausal change of coordinates can be found.

Example 13 *Consider the dynamical system described by the state-space representation*

$$\begin{array}{rcl} \dot{x}_1 & = & x_1(t) + x_2(t-2) + x_1(t-3)^2, \\ \dot{x}_2 & = & 2x_2(t) + x_1(t) + x_1(t-3)\,\mathrm{u}(t-1) - 2\left(x_2(t-3) + x_1(t-4)^2\right)x_1(t-1), \\ y(t) & = & x_1(t-3). \end{array} \tag{152}$$

First the matrix $A(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)$ is given by

$$\left.\begin{pmatrix} dy^{(2)} \\ d\dot{y} \\ dy \\ d\dot{u} \\ du \end{pmatrix}\right|_{y=H(\mathbf{x}_{[s]})} = \begin{pmatrix} \mathrm{u}(t-6)\,\delta^8 + 6x_1(t-6)\,\delta^6 + \delta^5 + \delta^3 & 3\delta^5 & x_1(t-8)\,\delta^6 & 0 \\ 2x_1(t-6)\,\delta^6 + \delta^3 & \delta^5 & 0 & 0 \\ \delta^3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx_1(t) \\ dx_2(t) \\ d\dot{u} \\ du \end{pmatrix}.$$

The left annihilator of the matrix $A(\mathbf{x}_{[s]}, \mathbf{u}_{[s]}, \delta)$ is given by the normalized vector

$$\lambda|_{y=H(\mathbf{x}_{[s]})} = \left(1 \vdots \;\; -\mathrm{u}(t-6)\,\delta^5 - \delta^2 + 2 \vdots \;\; -3 \vdots \;\; -x_1(t-8)\,\delta^6 \vdots \;\; 0\right).$$

Using the Algorithm 3 it is possible to compute (condition (i))

$$\begin{array}{rclcl} dh_0 & = & dy(t) & = & \delta^3 dx_1(t) \\ dh_1 & = & d\dot{y}(t) - d\Phi_1 & = & \left(\delta^3 + 2x_1(t-6)\delta^6\right) dx_1(t) + \delta^5 dx_2(t) + 3dx_1(t) \end{array} \tag{153}$$

which defines, as required by the condition (ii) of the Proposition 33, $P(\mathbf{x}_{[s]}, \delta)dx = Q(\delta)T(\mathbf{x}_{[s]}, \delta)dx$ expressed as

$$dh = \begin{pmatrix} \delta^3 & 0 \\ \left(3 + \delta^3 + 2x_1(t-6)\delta^6\right) & \delta^5 \end{pmatrix} dx(t) = \begin{pmatrix} \delta^3 & 0 \\ 3+\delta^3 & \delta^5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2x_1(t-1)\delta & 1 \end{pmatrix} dx(t). \tag{154}$$

This define the change of coordinates characterized by

$$T(\mathbf{x}_{[s]}, \delta)dx(t) = \begin{pmatrix} 1 & 0 \\ 2x_1(t-1)\delta & 1 \end{pmatrix} dx(t) = d\begin{pmatrix} z_1(t) \\ z_2(t) + z_1(t-1)^2 \end{pmatrix}, \tag{155}$$

That leads to the system in the new coordinate system expressed now as

$$\begin{array}{rcl} \dot{z}_1 & = & z_1(t) + z_2(t-2) \\ \dot{z}_2 & = & z_1(t) + 2z_2(t) + z_1(t-3)u(t-1), \\ y(t) & = & z_1(t-3). \end{array} \tag{156}$$

From the system (156) *and the deffinition given by the equation* (146) *we get* $d\bar{f}_1(\mathbf{z}_{[s]}, 0) = dz_1(t) + dz_2(t-2)$*, and* $d\bar{f}_2(\mathbf{z}_{[s]}, 0) = dz_1(t) + 2dz_2(t)$*. Performing the wedge product with the delayed output, and* $d\bar{f}_1(\mathbf{z}_{[s]}, 0),\ i = 1, 2\ d\bar{f}_1(\mathbf{z}_{[s]}, 0) \wedge dy(t) \wedge dy(t-1) \wedge dy(t-2) \wedge dy(t-3) = \left(dz_1(t) + dz_2(t-2)\right) \wedge dy(t) \wedge dy(t-1) \wedge dy(t-2) \wedge dy(t-3)$ *and* $d\bar{f}_2(\mathbf{z}_{[s]}, 0) \wedge dy(t) \wedge dy(t-1) \wedge dy(t-2) \wedge dy(t-3) = \left(dz_1(t) + 2dz_2(t)\right) \wedge dy(t) \wedge dy(t-1) \wedge dy(t-2) \wedge dy(t-3)$ *the vector* $d\bar{h}$*, defined by* $d\bar{h}_1 = dz_1(t) + dz_2(t-2)$ *and* $d\bar{h}_2 = dz_1(t) + 2dz_2(t)$*, is computed.*

Finally,

$$d\big(\bar{\phi}(\mathbf{z}_{[s]},\mathbf{u}_{[s]})\big) = d\dot{z}(t) - d\bar{h}(t) = \begin{pmatrix} 0 \\ z_1(t-3)du(t-1) + u(t-1)dz_1(t-3) \end{pmatrix}, \\ = \begin{pmatrix} 0 \\ d\left(z_1(t-3)u(t-1)\right) \end{pmatrix} = \begin{pmatrix} 0 \\ d\left(y(t)u(t-1)\right) \end{pmatrix}. \tag{157}$$

From (157) *it is straightforward to check that condition* (iv) *of the Proposition 33 is fulfilled. This means that the system defined by the equation* (152) *is linearizable via input-output injection.* ◁

Example 14 *Consider System* (123) *from Example 9. In example 10, conditions of Theorem 31 are used to find a non bicausal transformation to a system of the canonical form* (95)*. Nevertheless, note that the use of this non bicausal change of coordinates allows to compute the delayed state only. Now, by means of Proposition* (33) *it is possible to prove that a bicausal change of coordinates takes System* (123) *into the form* (95)*. This is done be rewriting* (134) *in the following way:*

$$\begin{pmatrix} dh_1 \\ dh_2 \end{pmatrix} = \begin{pmatrix} \delta^3 & 0 \\ 0 & \delta^3 \end{pmatrix} \begin{pmatrix} 1 & -2x_2(t-1)\delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = Q(\delta)P(\mathbf{x}_{[\mathbf{s}]},\delta)d\mathbf{x} \tag{158}$$

$P(\mathbf{x}_{[\mathbf{s}]},\delta)d\mathbf{x}$ *defines the bicausal change of coordinates given by the equations*

$$\begin{aligned} z_1(t) &= x_1(t) - x_2^2(t-1), & x_1(t) &= z_1(t) + z_2(t-1), \\ z_2(t) &= x_2(t), & x_2(t) &= z_2(t), \end{aligned} \tag{159}$$

that takes system into the form

$$\begin{aligned} \dot{z}_1(t) &= z_2\left(t\right), \\ \dot{z}_2(t) &= -a_0 z_1\left(t\right) - a_1 z_1\left(t-3\right) + z_1\left(t-3\right) z_1\left(t-4\right) + u(t), \\ y(t) &= z_1(t-3). \end{aligned} \tag{160}$$

In the new coordinates is possible to design a Luenberger observer of the form (96) *as*

$$\begin{aligned}\dot{\zeta}_1(t) &= \zeta_2(t) + k_1\big(\zeta_1(t-3) - z_1(t-3)\big),\\ \dot{\zeta}_2(t) &= -a_0\zeta_1(t) + k_2\big(\zeta_1(t-3) - z_1(t-3)\big) - a_1 y(t) + y(t)y(t-1) + u(t),\end{aligned} \tag{161}$$

which, defining an error equation $e(t) = z(t) - \zeta(t)$*, leads to the error time-delay equations*

$$\begin{aligned}\dot{e}_1(t) &= e_2(t) + k_1 e_1(t-3),\\ \dot{e}_2(t) &= -a_0 e_1(t) + k_2 e_1(t-3).\end{aligned} \tag{162}$$

A proper selection of k_1*, and* k_2 *lead to a stable solution of* (162) *that converges to the solution of* (161)*. The observer in the coordinates* $\xi_1(t) = \zeta_1(t) + \zeta_2^2(t-1)$*,* $\xi_2(t) = \zeta_2(t)$ *is expressed as*

$$\begin{aligned}\dot{\xi}_1(t) &= \xi_2(t) + k_1\big(\xi_1(t-3) - \xi_2^2(t-4) - x_1(t-3) + x_2^2(t-4)\big) +\\ &\quad 2\xi_2(t-1)\big(-a_0\big(\xi_1(t) - \xi_2^2(t-1)\big) - a_1 y(t-1) + y(t-1)y(t-2) +\\ &\quad k_2\big(\xi_1(t-4) - \xi_2^2(t-5) - x_1(t-4) + x_2^2(t-5)\big) + u(t-1)\big)\\ \dot{\xi}_2(t) &= -a_0\big(\xi_1(t) - \xi_2^2(t-1)\big) - a_1 y(t) + y(t)y(t-1) + u(t) +\\ &\quad k_2\big(\xi_1(t-3) - \xi_2^2(t-4) - x_1(t-3) + x_2^2(t-4)\big).\end{aligned} \tag{163}$$

The linearized dynamics (161) *allows to define a control law* $u(t) = -(\bar{k}_1 - a_0)\big(\xi_1(t) + \xi_2^2(t-1)\big) - \bar{k}_2\xi_2(t) + a_1\big(\xi_1(t-3) + \xi_2^2(t-4)\big) - \big(\xi_1(t-3) + \xi_2^2(t-4)\big)\big(\xi_1(t-4) + \xi_2^2(t-5)\big)$*. Using such a control feedback in Figure 24 the numerical solution of the of the closed loop system is presented considering the system parameters* $a_0 = 5$*,* $a_1 = -3$*, a time delay of* $1/3$*, observer coefficient values* $k_1 = 1.2$ *and* $k_2 = 3$*, and the values for the controller coefficients chosen as* $\bar{k}_1 = 14.14$ *and* $\bar{k}_2 = 10$*.*

◀

5.5 Discussion.

The linearization results, reported in García-Ramírez *et al.* (2016), define a constructive solution for the invertible transformation to the canonical form (95). Throughout the chapter, results for the effective computation complements the conditions given. Also, we give conditions that ensure that, after the change of coordinates, the system is represented

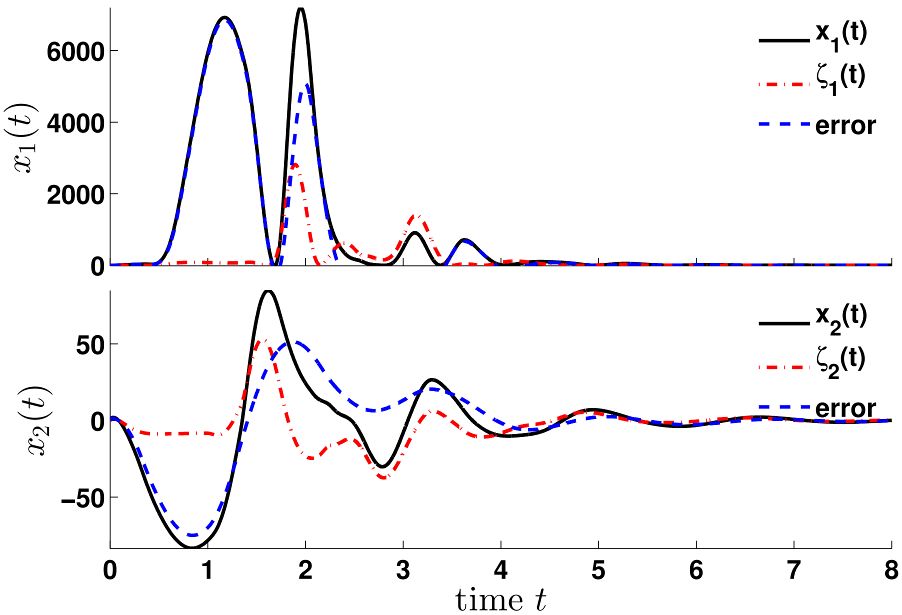


Figure 24: Observation of the NLDS (123)**.**

in the linearized form (95). An academic example is proposed to show the computations required for the linearization and the observer design. Numerical results are reported to show the performance of the designed observer. The observed signals used as a feedback were successful stabilizing the system. It is important to point out that the syntonization of both, observer and controller, is a topic outside of the scope of this work, so the efficiency of the response is not evaluated. Moreover, observations should be considered about the resultant linear equations in Examples 10, and 14. Note that, after the elimination of the nonlinear part, (135) is a delay-free system, while (161) is affected by the time-lag. This allowed to use delay-free strategies on the error dynamics (137) for the selection of the gain for the observer, and reserved the observer design for the time-delayed error differential equation (162) to time-delay techniques. A strong restriction consists in the impossibility to change the parameter a_0. Therefore, for values outside of $0 < \sqrt{a_0\tau^2} < \pi$, there does not exist a value k_2 that fulfills Theorem 1 for the asymptotic stability of the error dynamics (162), for $k_1 = 0$. This means that in such a case Lemma 9 is not applicable. Also, the error of the observed states computed by observer (138) converges to zero faster than observer (163), which computes the present state. This might be because the criteria for the restrictions presented for the observer design on an aftereffect error dynamics as (162). Moreover, since the initial conditions of the system are not provided to the observer dynamics, the observation error is considerably big until the information of the system is injected on the observer. This should be considered in unstable systems with fast rates of change since observations might not be fast enough to stabilize the system. The study of this results on a quarter-car dynamical system was reported in Garcia-Ramirez *et al.* (2016).

Chapter 6. Conclusions and Pespectives

The main contributions of this work are addressed to the problem of simplification of hereditary dynamical systems via coordinate transformations and approximation by delay-free equations. The conclusions and perspectives, enlisted in the order of the methodology proposed, are summarized as follows.

The solution presented for the problem of linearization , via elimination of the injection function, by means of an invertible change of coordinates is constructive allowing to implement an algorithm which permits to compute explicitly the transformation and the injection function. Moreover, since the algorithm performs its computations step by step, using the state-space variables, its implementation using computational software is straightforward. The extension of this solution to the case where the coordinate transformation is bicausal ensures also the causality in the design of control laws or state-space observers. Nevertheless, as was pointed out by examples, the use of bicausal transformations may lead to representations where the injection functions depend on future values of the output variable. Despite this, an observer that computes the delayed state can be designed using the results given using an invertible change of coordinates. Furthermore, numerical results of an academical example are presented to show the effectiveness of a Luenberger-type observer designed based on the canonical form given by this strategy. This example illustrates how the observed variables can be effectively used to stabilized a nonlinear system, and how the knowledge of the observed state variable in the present time permits to design a control input using delay-free techniques. Further investigations on the controller and observer syntonization might improve the performance of the technique and it is left for future work.

The importance of the geometrical control approach, particularly in the solution of problems of equivalence with canonical forms, motivated the study of the extension of the Lie bracket operation recently developed for the time-delay system framework. The algorithm developed in this work of thesis simplify the procedure needed in the solution of problems that involves this important operation. The computation time is reduced in the iterative procedures as the required in the test of equivalence with linear systems, the existence of an integrable left-annihilator of a submodule with elements in $\mathcal{K}^n(\delta]$, among others (see Table 1 in page 15). This is a direct consequence of the reduction of a numerous amount of computations involved in the direct calculation of the extended Lie bracket. Future work on the implementation of the above results in symbolic computational software (see Gárate-García *et al.* (2011)) is a direct consequence of the constructive solutions proposed throughout this work. It is notable that the algorithm presented provides a structure that allows the implementation using parallel processing techniques. Such techniques are capable of a considerable computational time reduction. An implementation in a software environment permits to take advantage of results produced in this thesis, and in other works with a reduced inversion of training and computational time.

The linearization results involving coordinate transformations allows to treat locally a family of time-delay nonlinear systems as, not necessary delay-free, linear systems. Nevertheless, to deal with delay-free differential equations brings advantages considering the wide developed theory of this kind of systems. A well-known strategy to eliminate the aftereffect on the equations is the approximation by Taylor series of the delayed variables. In this work, general conditions that ensure the stability of the ordinary differential equations that results from the Taylor series approximation of the delayed variable are presented. These conditions show that, after the approximation, the resulting ordinary differential equation is unstable, despite the stability of the original time-delay system, for a certain truncation order. Even more, numerical results indicate that, for small values of the truncation, the approximation is not accurate. However, as a result of this thesis, it is proved that, for a class of linear delay differential equations, feasible approximations can be characterized by forward-shifting the time, and substituting the delayed and advanced variables by the truncated Taylor series. This approximation takes account of the stability regions, in the parametric space, of the studied systems. Nevertheless, since the results in this thesis are restricted to the mentioned equations, a problem stated for future work is to investigate if the methodology used can be extended to the general case. Another limitation of this strategy is that it is limited to those families of hereditary linear systems where stability regions are defined. Nonetheless, a direct application of this result is in the selection of coefficients for the stabilization of linear time-delay control systems.

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