Dynamics of Atmospheres and Oceans 75 (2016) 46–57

Contents lists available at ScienceDirect

Dynamics of Atmospheres and Oceans

journal homepage: www.elsevier.com/locate/dynatmoce

Short Communication

Free and forced Rossby normal modes in a rectangular gulf of arbitrary orientation

Federico Graef[*,1]

Centro de Investigación Científica y de Educación Superior de Ensenada (CICESE), Ensenada, B.C., Mexico

A R T I C L E I N F O

Article history:
Received 14 December 2015
Received in revised form 19 May 2016
Accepted 24 May 2016
Available online 27 May 2016

Keywords:
Planetary waves
Gulf
Modes
Solutions

A B S T R A C T

A free Rossby normal mode in a rectangular gulf of arbitrary orientation is constructed by considering the reflection of a Rossby mode in a channel at the head of the gulf. Therefore, it is the superposition of four Rossby waves in an otherwise unbounded ocean with the same frequency and wavenumbers perpendicular to the gulf axis whose difference is equal to $2m\pi/W$, where m is a positive integer and W the gulf's width. The lower (or higher) modes with small m (or large m) are oscillatory (evanescent) in the coordinate along the gulf; these are elucidated geometrically. However for oceanographically realistic parameter values, most of the modes are evanescent.

When the gulf is forced *at the mouth* with a single Fourier component, the response is in general an infinite sum of modes that are needed to match the value of the streamfunction at the gulf's entrance. The dominant mode of the response is the resonant one, which corresponds to forcing with a frequency ω and wavenumber normal to the gulf axis η appropriate to a gulf mode: $\eta = -\beta \sin\alpha/(2\omega) \pm M\pi/W$, where α is the angle between the gulf's axis and the eastern direction (+ve clockwise) and M the resonant's mode number. For zonal gulfs ω drops out of the resonance condition.

For the special cases $\eta = 0$ in which the free surface goes up and down at the mouth with no flow through it, or a flow with a sinusoidal profile, resonant modes can get excited for very specific frequencies (only for non-zonal gulfs in the $\eta = 0$ case). The resonant mode is around the annual frequency for a wide range of gulf orientations $\alpha \in [40^\circ, 130^\circ]$ or $\alpha \in [220^\circ, 310^\circ]$ and gulf widths between 150 and 200 km; these include the Gulf of California and the Adriatic Sea. If η is imaginary, i.e. a flow with an exponential profile, there is no resonance. In general less modes get excited if the gulf is zonally oriented.

1. Introduction

In this note we study the Rossby normal modes in a mid-latitude β-plane gulf of arbitrary orientation from a theoretical point of view. Beyond the value of advancing our knowledge in GFD, this study is also motivated by the vast literature devoted to the study of gulfs (mainly observational and numerical). Several studies (for example, Ripa, 1990, 1997) showed that the Gulf of California is forced at the mouth by an annual baroclinic Kelvin wave. Beier (1997) studied the dynamics of this gulf with a horizontal two-dimensional linear two-layer numerical model, mentioning that the β-effect has little influence

* Correspondence to: CICESE, P.O. Box 434844, San Diego, CA 92143-4844, United States. Tel.: +52 646 175 05 00.
E-mail address: fgraef@cicese.mx

[1] Currently on sabbatical at the International Centre for Theoretical Physics (ICTP), Trieste, Italy.

http://dx.doi.org/10.1016/j.dynatmoce.2016.05.005
0377-0265/

in the dynamics and that annual long Rossby waves do not fit in the gulf (average width of 150 km) because they have a zonal wavelength of ~650 km (calculated from the long Rossby wave dispersion relation and using a value of 30 km for the baroclinic Rossby radius); and that short Rossby waves will be subject to dissipation. Does this mean that there cannot be Rossby modes in the Gulf of California? In this paper we will show that for a gulf with the orientation and width of the Gulf of California, the $M=2$ resonant mode is around the annual frequency for a forcing at the mouth in which the free surface just goes up and down. In a review paper Lluch-Cota et al. (2010) discussed how the Gulf of California influences the northward propagation of coastal trapped Kelvin waves associated with El Niño (ENSO) events, and how this signal results in an ENSO signature inside the Gulf. An open question in the above studies is to find out what the QG response is (given in terms of Rossby gulf modes) to this type of Kelvin wave forcing at the mouth.

As regards the Gulf of Mexico, Oey and Lee (2002) and Hamilton (2009) showed that deep eddy energy can be explained by topographic Rossby waves. Rossby wave theory has been used to study the dynamics of the Loop Current and the shedding of eddies (Hurlburt and Thompson, 1982) in a numerical model. These are examples that show the importance of Rossby waves in the dynamics of this gulf.

There are very few references about Rossby waves in the Adriatic Sea and they refer to topographic waves. According to Pasarić et al. (2000), the propagation of topographic Rossby waves within the basin, although related to small surface displacements, could perhaps influence the adjustment at the highest frequencies they considered.

Although not strictly gulfs (they are channels), there have been some studies of Rossby waves in the South China Sea (SCS) and in the Mozambique Channel. Shu et al. (2016) observed energetic fluctuations below 1400 m from direct current measurements in the SCS that are attributable to topographic Rossby waves. Wu et al. (2008) observed free and forced Rossby Waves in the western SCS inferred from Jason-1 satellite altimetry data while Yang and Liu (2003) interpreted sea surface height anomalies in terms of forced annual Rossby waves in the northern SCS. Harlander et al. (2009) showed that a westward-propagating signal observed in the flow through the channel could be a Mozambique channel Rossby normal mode (a meridional channel mode independent of the N–S coordinate and thus with a velocity parallel to the channel only) with a period of 70 days. Like in many other papers, these are examples illustrating the importance of Rossby modes to explain observations.

Of the very few theoretical studies about Rossby normal modes in a gulf we can cite García and Graef (1998), who analysed the nonlinear self-interaction of one of such modes.

Questions like: How are the Rossby modes in a gulf?, What modes are excited when the gulf is forced at the mouth?, Are there resonant modes?, What are the frequencies and wavenumbers of these modes?, And so on are the subject of this paper. The answers to these should be of interest not only from the point of view of GFD but could also help explain the observations and definitely serve as a tool for numerical modelling efforts.

In the next section, we compute the Rossby normal modes in an idealised gulf: a parallelepiped or a rectangular box which allows analytical treatment. A Rossby gulf mode is constructed by considering the reflection of a channel mode at the gulf's head and a graphical method to find gulf modes is provided, paying careful attention to the cases when the wavenumbers parallel to the gulf's axis are complex or equal. A graph of the dispersion relation is also included. Then in Section 3 we find the Rossby gulf modes that get excited when we force the gulf at the mouth with a single Fourier component of a fixed frequency and wavenumber perpendicular to the gulf's axis. Two cases are distinguished: resonant and non-resonant forcing. We show contour maps of the resonance condition spanning a wide range of mode and physical parameters and some examples of the forced solution. Also, three special cases of forcing wavenumbers perpendicular to the gulf's axis are studied that correspond to three distinctively physical mechanisms: one in which there is no flow at the mouth, a second one in which there is a net flow through the mouth (but zero over one period) with an exponential profile resembling a Kelvin wave and a third in which there is inflow and outflow but with zero net flow at all times. We end the paper with discussion and conclusions.

2. Free Rossby modes in a gulf

Consider a β-plane with a coordinate system (x, y, z) in which x is parallel, y perpendicular to the gulf and z vertically upwards. The gulf has length L, width W, a flat bottom and it is oriented such that its axis makes an angle α with the circles of latitude (positive clockwise), see Fig. 1. To simplify matters without sacrificing the essentials, we consider a barotropic ocean with a free surface.

For the gulfs that we will be considering later for the numerical calculations of the analytical results like the Gulf of California, the Adriatic Sea or the Red Sea with their north–south extension spanning several degrees of latitude (around 10, 6 and 15, respectively), the β-plane would be more suitable than the f-plane. Besides this, we need $\beta \neq 0$, otherwise the Rossby modes will be time-independent (zero frequency).

The governing equation is the linear quasigeostrophic (QG) potential vorticity equation:

$$\partial_t \left(\nabla^2 - r_d^{-2}\right) \psi + \beta(\cos\alpha\, \partial_x + \sin\alpha\, \partial_y)\psi = 0, \tag{1}$$

where $\nabla^2 = \partial_x\partial_x + \partial_y\partial_y$, t is time, ψ the QG streamfunction, β the northward gradient of the planetary vorticity and $r_d = (gH)^{1/2}/f_0$ the barotropic Rossby radius in which g is the acceleration of gravity, H the depth of the ocean and f_0 the Coriolis parameter.

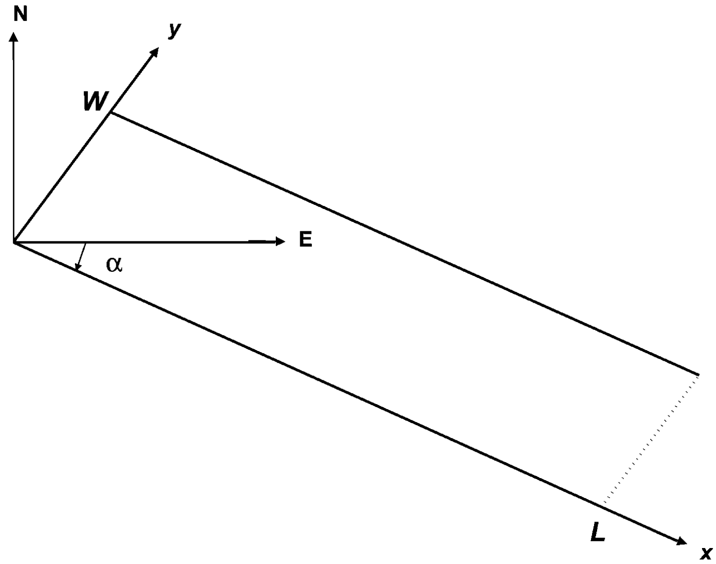

Fig. 1. Coordinate system and gulf geometry. The rotated coordinate system has x parallel and y perpendicular to the gulf, which has length L and width W. The gulf is oriented such that its axis makes an angle α with the eastern direction (positive clockwise). There is an open boundary at $x = L$.

The boundary conditions of no normal flow at the lateral solid walls are $\partial_x \psi = 0$ at $y = 0$, W and $\partial_y \psi = 0$ at $x = 0$, where the gulf's head is located.

There is an open boundary at the gulf's mouth, $x = L$, where the normal velocity is not necessarily zero. Since the domain is partially open, an explicit mass conservation constraint or time independent circulation is not required (Pinardi and Milliff, 1990). Thus, it suffices to simply set $\psi = 0$ at the solid walls, i.e. at the gulf's perimeter.

We can construct a Rossby mode in a gulf by considering the reflection of a *channel mode* at the head of the gulf, i.e. at $x = 0$. A channel mode can be written as (Graef and Müller, 1996)

$$\psi_m = \mathrm{Re}\left\{ a_m \exp\left[i\left(kx - \frac{\beta \sin\alpha}{2\omega} y - \omega_m t \right)\right] \sin\left(\frac{m\pi y}{W}\right)\right\}, \quad m = 1, 2, 3, \ldots, \tag{2}$$

where the following dispersion relation holds:

$$\left(k + \frac{\beta \cos\alpha}{2\omega}\right)^2 + \frac{m^2\pi^2}{W^2} = \frac{\beta^2}{4\omega^2} - \frac{1}{r_d^2}. \tag{3}$$

From this we see that there are in general two roots for k, call them $k_1 = k_+$ and $k_2 = k_-$. Geometrically, these two roots are the k-coordinates or abscissas of the intersections of the lines parallel to the channel (in wavenumber space) given by $l_{1,2} = -\beta \sin\alpha/(2\omega) \pm m\pi/W$ with the slowness circle ω = constant if the roots are real; four intersections for each m (see Fig. 2). If the roots are complex (complex conjugates) it is $k_1 = k_2^*$ with $k_{1,2} = k_r \pm i k_i$, whose real part is $k_r = -\beta\cos\alpha/(2\omega)$ and whose imaginary part satisfies an equation of the hyperbola for ω = constant:

$$k_i^2 = \left(l + \frac{\beta \sin\alpha}{2\omega}\right)^2 + \frac{1}{r_d^2} - \frac{\beta^2}{4\omega^2}. \tag{4}$$

Therefore for each mode number m, the line l_1 (l_2) parallel to the channel intersects the large l (small l) branch of the hyperbola given by (4) at two points. Note that the larger the mode number m is, the farther away from the origin these intersections would be (see Fig. 2). When $k_{1,2}$ are complex, it is $l < -\beta\sin\alpha/(2\omega) - R$ or $l > -\beta\sin\alpha/(2\omega) + R$, where $R = \left[\beta^2/(4\omega^2) - r_d^{-2}\right]^{1/2}$ is the radius of the slowness circle. For any l in either of these open intervals, the corresponding two points on the hyperbola $(\pm k_i, l)$ define an exponentially decaying (k_1)–exponentially growing (k_2) Rossby wave pair with respect to the gulf's head at $x = 0$. The two branches of the hyperbolae have transverse axis coinciding with the l axis and vertices at $[0, -\beta\sin\alpha/(2\omega) \pm R]$. In Fig. 2 we show the slowness circle, the hyperbolae and the intersections of these curves with the lines $l_{1,2}$ for a given ω, gulf orientation α, width W and depth H that could resemble the Gulf of California and for the first five modes. The first two $m = 1, 2$ have k real and the rest $m > 2$ have k complex.

These four intersections of $l_{1,2}$ with either the circle or the hyperbola (for given ω and m) are indeed the four Rossby waves that conform the gulf's mode. One channel mode (the incident) has wavenumber vectors $[k_1, -\beta\sin\alpha/(2\omega) \pm m\pi/W]$; the other channel mode (the reflected) has wavenumber vectors $[k_2, -\beta\sin\alpha/(2\omega) \pm m\pi/W]$. It should be noted that this is true *for all* orientations of the gulf (see Graef and Magaard, 1994). If $k_{1,2}$ are complex, the incident channel mode becomes an exponentially decaying mode (in x), whereas the reflected mode becomes exponentially growing, both having an oscillatory part $\propto e^{i k_r x}$.

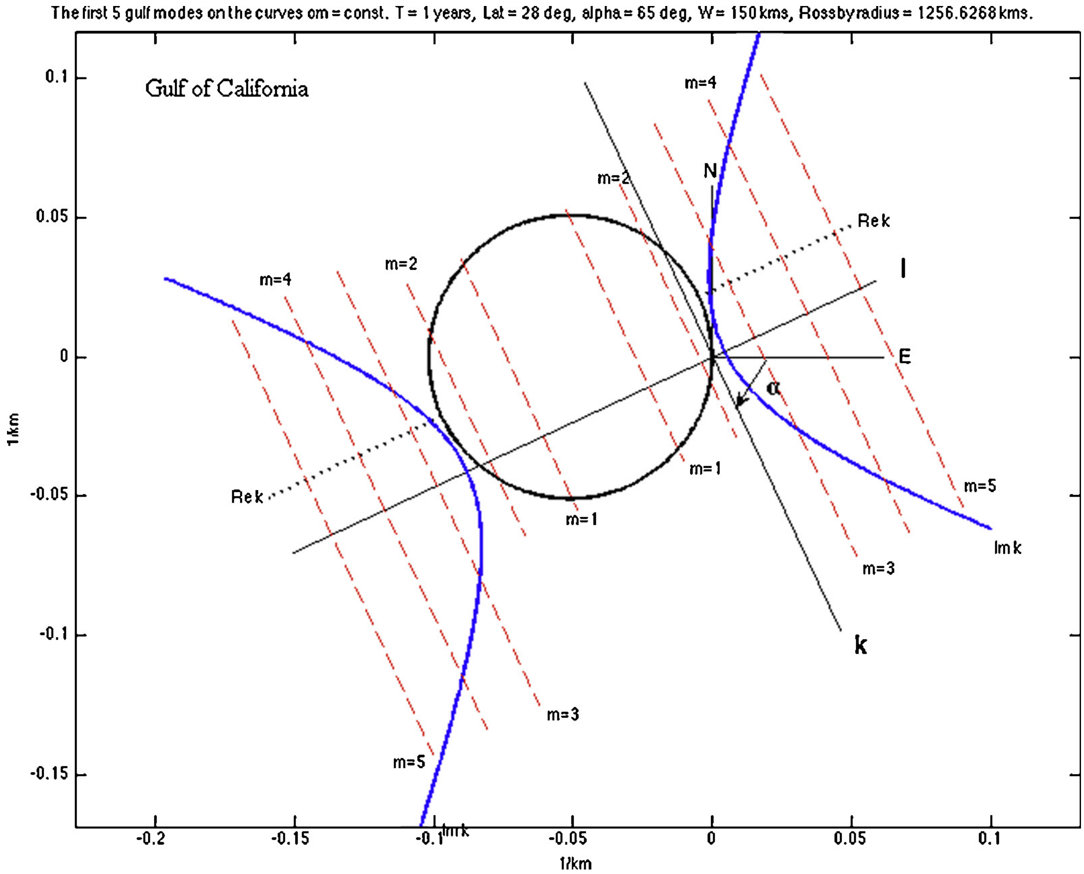

Fig. 2. Graphical representation in wavenumber space of a Rossby normal mode in a gulf. For given frequency ω and mode number m, the four intersections of $l_{1,2} = -\beta \sin\alpha/(2\omega) \pm m\pi/W$ (red dashed lines) with either the circle in black (for k real) or the hyperbolae in blue (for k complex) are the four Rossby waves that conform the gulf mode. The gulf mode may also be viewed as the sum of two channel modes: the incident (with respect of the gulf's head) that has wavenumber vectors $[k_1, -\beta \sin\alpha/(2\omega) \pm m\pi/W]$ and the reflected with wavenumber vectors $[k_2, -\beta \sin\alpha/(2\omega) \pm m\pi/W]$. Shown in the graph are the first five gulf modes: $m = 1, 2$, which have k real, and $m = 3, 4, 5$ which have k complex; k_r is also shown (dotted black line). Parameters resembling the Gulf of California: reference latitude = 27.5° N, $\alpha = 65°$, $W = 150$ km, $H = 730$ m, yielding $r_d = 1257$ km; $\omega = 2\pi/(1$ year). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The solution for the gulf (i.e. a Rossby mode of a gulf) is simply the superposition of these two channel modes such that $\psi = 0$ at $x = 0$. This amounts to replace e^{ikx} in (2) by $e^{ik_1x} - e^{ik_2x}$, and after some trigonometric identities the gulf mode can be written as

$$\psi = \text{Re}\left\{a_m \exp\left[-i\frac{\beta}{2\omega}(x\cos\alpha + y\sin\alpha) - i\omega t\right]\sin(\delta_m x)\sin\left(\frac{m\pi y}{W}\right)\right\}, \quad m = 1, 2, 3, \ldots, \tag{5}$$

where

$$\delta_m = \left(\frac{\beta^2}{4\omega^2} - \frac{1}{r_d^2} - \frac{m^2\pi^2}{W^2}\right)^{1/2}, \tag{6}$$

or, given m and δ, it is $\omega_m = (\beta/2)\left(\delta^2 + m^2\pi^2/W^2 + r_d^{-2}\right)^{-1/2}$. From (3) and (6) we easily see that $\delta_m = (k_1 - k_2)/2$. Note that in (5) the carrier wave always propagates westward with phase speed $C = 2\omega^2/\beta$, so there is no loss of generality by choosing the positive frequency. The higher modes (large m) will have δ_m imaginary, i.e. $k_{1,2}$ complex, and consequently they will be evanescent in x, that is $\propto \sinh(|\delta_m|x)$.

One last detail: if the roots $k_{1,2}$ are equal for a mode $m = M_e$ then $R = M_e\pi/W$. The two independent solutions are e^{ik_rx} and xe^{ik_rx}, but to satisfy the boundary condition at the head of the gulf ($x = 0$), we need to take the second one, so that in this case the x-dependence of the M_e mode is $\propto xe^{ik_rx}$, i.e. the mode grows linear in x. We will not deal specifically with this case in the foregoing analysis but it is important because it is part of the spectrum of free gulf Rossby modes, which can be either oscillatory in x, with an amplitude growing linearly in x or evanescent in x.

For realistic gulf dimensions (such that planetary wave motion could matter) and frequencies of Rossby wave motion, most of the modes are evanescent in x. In Fig. 3 we plot δ_m^2 versus the Rossby mode period $T = 2\pi/\omega$ from the dispersion relation (6) for several mode numbers and for two different Rossby radii of deformation (barotropic and baroclinic) and two

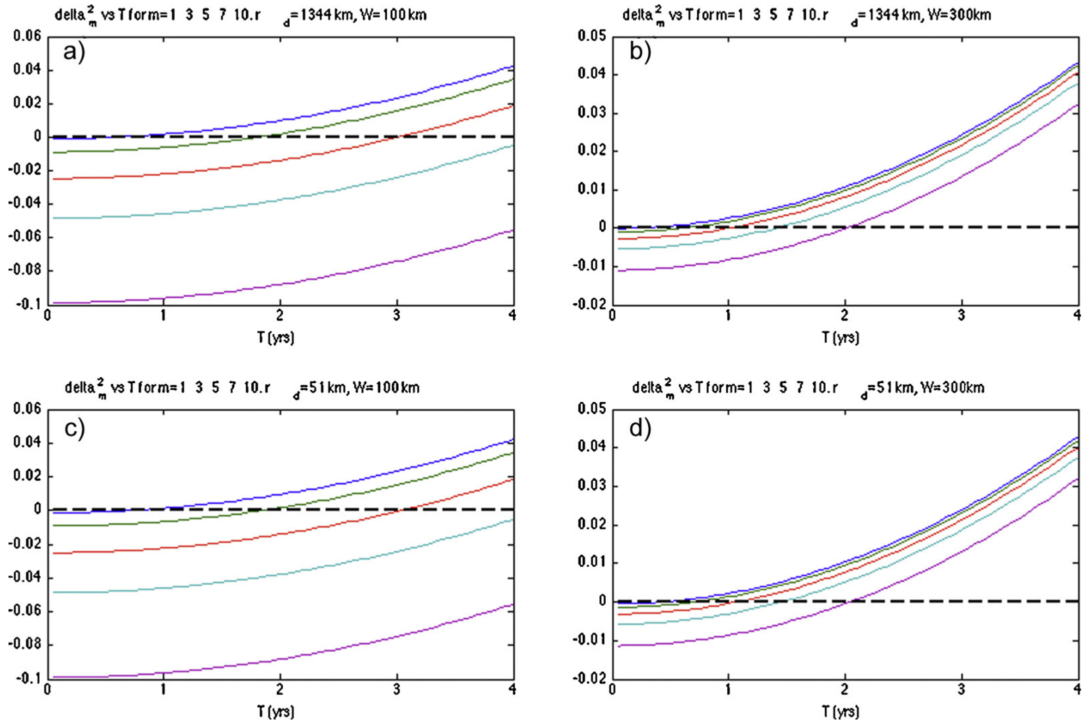

Fig. 3. The dispersion relation. Curves δ_m^2 (in km^{-2}) versus the Rossby mode period $T = 2\pi/\omega$ (in years) from the dispersion relation (6) for the mode numbers $m = 1$ (blue), $m = 3$ (green), $m = 5$ (red), $m = 7$ (cyan) and $m = 10$ (magenta). The four cases shown are for two different Rossby radii of deformation (barotropic and baroclinic) and two gulf widths: (a) $r_d = 1344$ km, $W = 100$ km; (b) $r_d = 1344$ km, $W = 300$ km; (c) $r_d = 51$ km, $W = 100$ km; (d) $r_d = 51$ km, $W = 300$ km. The $\delta_m^2 = 0$ line is indicated by a thick black dotted line. Every mode above (below) this line is oscillatory or periodic (evanescent) in x. Parameters: reference latitude = 25° N, ocean depth $H = 700$ m for the barotropic case and the equivalent depth for the baroclinic case $H_e = 1$ m.

gulf widths. Note that only the first few modes are oscillatory in x for periods less than a year or so; in fact for a barotropic gulf 100 km wide only the first mode is oscillatory in x for the annual frequency (not shown in graph). Of course for smaller frequencies (periods larger than one year) we get more oscillatory modes. That is, in general, for a large portion of parameter space $\delta_m^2 < 0$. In fact the sign of δ_m^2, which determines whether a mode is oscillatory o evanescent, only depends on the frequency, the ratio between the gulf's width and the Rossby radius $\gamma = W/r_d$, and of course the mode number m. This can be seen by non-dimensionalizing (6) using $\delta_m = (1/W)\delta'_m$ and $\omega = \beta W \omega'$, where prime denotes non-dimensional variables, which yields

$$\delta_m'^2 = \frac{1}{4\omega'^2} - \gamma^2 - m^2\pi^2. \tag{7}$$

3. Forcing at the mouth

A physically interesting and analytically tractable problem is when the gulf is forced at the mouth at a *fixed* frequency. Since we have a linear problem, without loosing generality we may consider a single Fourier component of the forcing of the form

$$\psi = Fe^{i(\eta y - \omega t)} \quad \text{at} \quad x = L, \tag{8}$$

where the constant F is the amplitude of the forcing and η is the wavenumber of the forcing.

To match the imposed forcing function at the mouth we need to add, in general, an infinite number of gulf modes (all having the same frequency of the forcing):

$$\exp\left[-i\frac{\beta}{2\omega}(L\cos\alpha + y\sin\alpha)\right]\sum_{m=1}^{\infty} a_m \sin(\delta_m L)\sin\frac{m\pi y}{W} = Fe^{i\eta y}, \tag{9}$$

where the mode amplitudes are obtained using the orthogonality of $\{\sin(m\pi y/W)|m = 1, 2, 3, \ldots\}$, which is a complete set in $[0, W]$, i.e.

$$a_m = \frac{2F}{W\sin(\delta_m L)}\exp\left(\frac{i\beta L\cos\alpha}{2\omega}\right)\int_0^W \exp\left(\frac{i\beta y\sin\alpha}{2\omega} + i\eta y\right)\sin\frac{m\pi y}{W}\,dy, \quad m = 1, 2, 3, \ldots. \tag{10}$$

To proceed, we need to compute the integral in (10) for the mode amplitudes. It is

$$I = \int_0^W \exp\left[\frac{i\beta y \sin\alpha}{(2\omega)} + i\eta y\right] \sin\frac{m\pi y}{W}\, dy = \left(\frac{m\pi}{W}\right) \frac{1 - e^{i[\beta\sin\alpha/(2\omega)+\eta]W}(-1)^m}{m^2\pi^2/W^2 - \left[\beta\sin\alpha/(2\omega)+\eta\right]^2}. \quad (11)$$

Now a little detour to give an overview on how to proceed in the baroclinic case: ψ given by (5) is multiplied by the nth baroclinic mode $\Psi_n(z)$, r_d is replaced by the nth mode baroclinic Rossby radius in (6) and the z-dependence of the forcing is projected on to the vertical modes $\{\Psi_n(z)|n = 0, 1, 2, \ldots\}$, which also form a complete set in $[-H, 0]$, being the eigenfunctions of a vertical Sturm–Liouville problem. Finally, to match the imposed forcing function at the mouth given by $e^{i\eta y}F(z)$, one writes the solution as a double sum:

$$\exp\left[-i\frac{\beta}{2\omega}(L\cos\alpha + y\sin\alpha)\right] \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin(\delta_{nm}L)\,\Psi_n(z)\,\sin\frac{m\pi y}{W} = e^{i\eta y}F(z). \quad (12)$$

3.1. Resonant forcing

First, we note that the denominator in (11) is zero if $\left[\beta\sin\alpha/(2\omega)+\eta\right]^2 = m^2\pi^2/W^2$. This could happen only for one m, call it $m = M$, because both ω and η, although free to be chosen, are fixed once they are selected. Therefore the denominator is zero if for some $m = M$,

$$\eta_\pm = -\frac{\beta\sin\alpha}{2\omega} \pm \frac{M\pi}{W}. \quad (13)$$

This is the condition for *resonance*. We are forcing at the mouth with the frequency ω and wavenumber component normal to the gulf's axis η appropriate to a gulf's mode [recall that $l_{1,2} = -\beta\sin\alpha/(2\omega) \pm m\pi/W$].

Now for this mode $m = M$, we need to re-calculate the integral, since (11) is no longer valid. The result is $I_M = \pm iW/2$ so that

$$a_M = \frac{\pm i F e^{i\beta L\cos\alpha/(2\omega)}}{\sin(\delta_M L)}. \quad (14)$$

For the rest of the amplitudes $m \neq M$, one simply substitutes (13) in (11) and gets

$$a_m = \frac{2Fm e^{i\beta L\cos\alpha/(2\omega)}\left[1-(-1)^{M+m}\right]}{\pi\,\sin(\delta_m L)\left(m^2 - M^2\right)} \quad (m \neq M). \quad (15)$$

Therefore if the resonant mode M is odd (even), all the rest of the odd (even) modes vanish.

The resonant mode $m = M$ is the dominant one because it has the largest amplitude; we have that

$$\left|\frac{a_m}{a_M}\right| = \frac{2\,m\left[1-(-1)^{M+m}\right]\sin(\delta_M L)}{\pi\left(m^2 - M^2\right)\sin(\delta_m L)}, \quad (16)$$

which is of the order of $1/m$ for the first oscillatory modes and of the order of $1/[m\sinh(m)]$ for the rest of the evanescent modes. Therefore, out of all the modes that are excited with the imposed forcing, the dominant mode, i.e. the one that exhibits the largest response, is the resonant mode. It is physically relevant and therefore it sufficiently justifies the analysis of these cases.

If the gulf is zonally oriented ($\sin\alpha = 0$) the condition for resonance reduces to $\eta_\pm = \pm M\pi/W$, i.e. irrespective of the frequency of the forcing.

The resonance condition (13) can be rewritten as

$$\eta_\pm W = -\frac{\beta WT\sin\alpha}{4\pi} \pm M\pi, \quad (17)$$

which is more suitable for plotting purposes. In Fig. 4 we plot contour maps of the non-dimensional quantity $\eta_\pm W$ as a function of the non-dimensional quantities βWT and α for three resonant mode numbers. For the chosen reference latitude of 35° N, $\beta WT = 59.14$ for $T = 1$ yr and $W = 100$ km, or any combination of $WT = 100$ km yr. Note that for values of $\beta WT < 40$ the term $\pm M\pi$ dominates; for larger values the first term dominates except near $\alpha = 0°$, 180°, where $\eta_\pm W$ becomes the constant $\pm M\pi$, a horizontal contour line. Graphically a resonant mode is given by the value of the contour (the value of $\eta_\pm W$) intersected by the vertical line βWT = constant (which in turn represents an infinite number of combinations of W's and T's) and the horizontal line α = constant (a value of the gulf orientation); one intersection on the left panel and one on the right for the corresponding M.

For zonally oriented gulfs ($\sin\alpha = 0$) the resonant frequency could be any value, as shown by the horizontal contour lines in Fig. 4 for $\alpha = 0°$, 180°.

A contour plot of the solution ψ at 4 different times (separated one quarter of a period) is shown in Figs. 5 and 6 for the case in which the resonant mode ($M = 1$) is oscillatory in x and in which the resonant mode ($M = 3$) is evanescent in x,

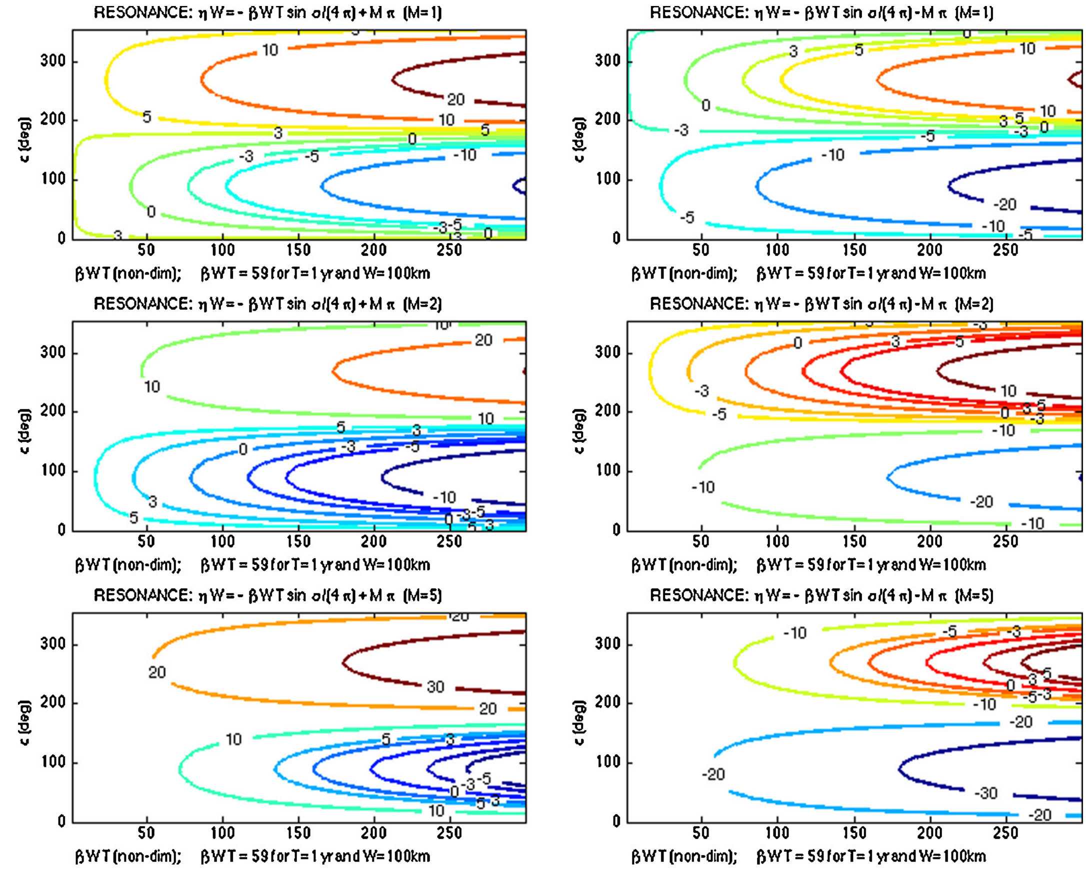

Fig. 4. The resonance condition. Contour maps of the non-dimensional quantity $\eta_{\pm}W$, i.e. the product of the wavenumber of the resonant forcing and the gulf's width, as a function of the non-dimensional quantities βWT and α for three resonant mode numbers as given by the resonance condition (17). (a) $\eta_{+}W$, $M=1$; (b) $\eta_{-}W$, $M=1$; (c) $\eta_{+}W$, $M=2$; (d) $\eta_{-}W$, $M=2$; (e) $\eta_{+}W$, $M=5$; (f) $\eta_{-}W$, $M=5$. Reference latitude = 35° N. For $T=1$ yr and $W=100$ km, or any combination of $WT=100$ km yr, $\beta WT=59.14$.

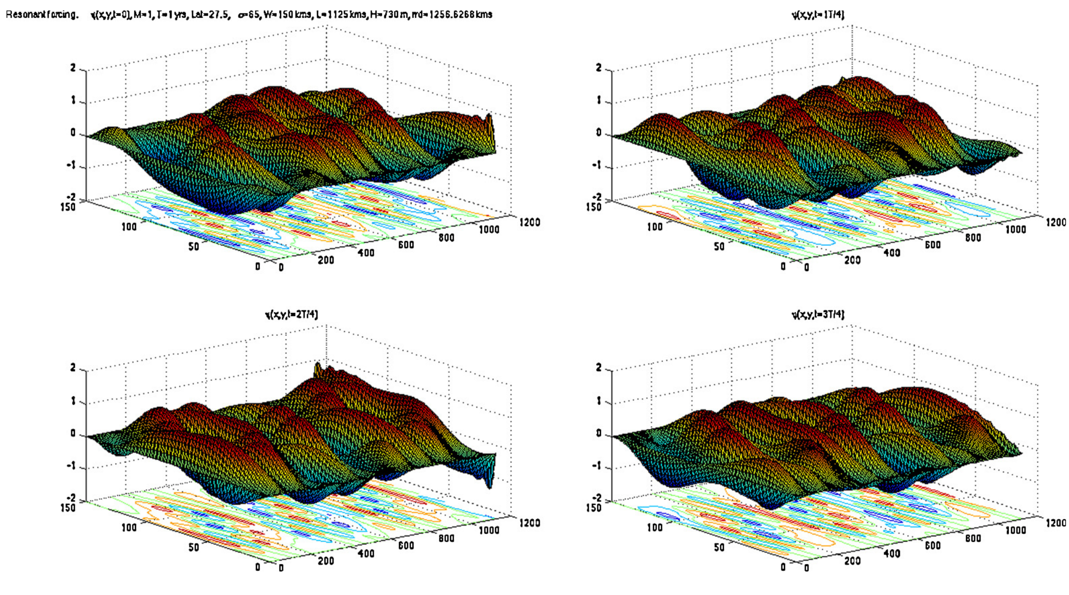

Fig. 5. $\psi(x, y, t)$ at $t = 0, T/4, T/2, 3T/4$ for a resonant mode $M=1$ (oscillatory). Parameters of Gulf of California: reference latitude = 27.5° N, $\alpha=65°$, $W=150$ km, $L=1125$ km, $H=730$ m $\rightarrow r_d=1257$ km. $T=1$ yr.

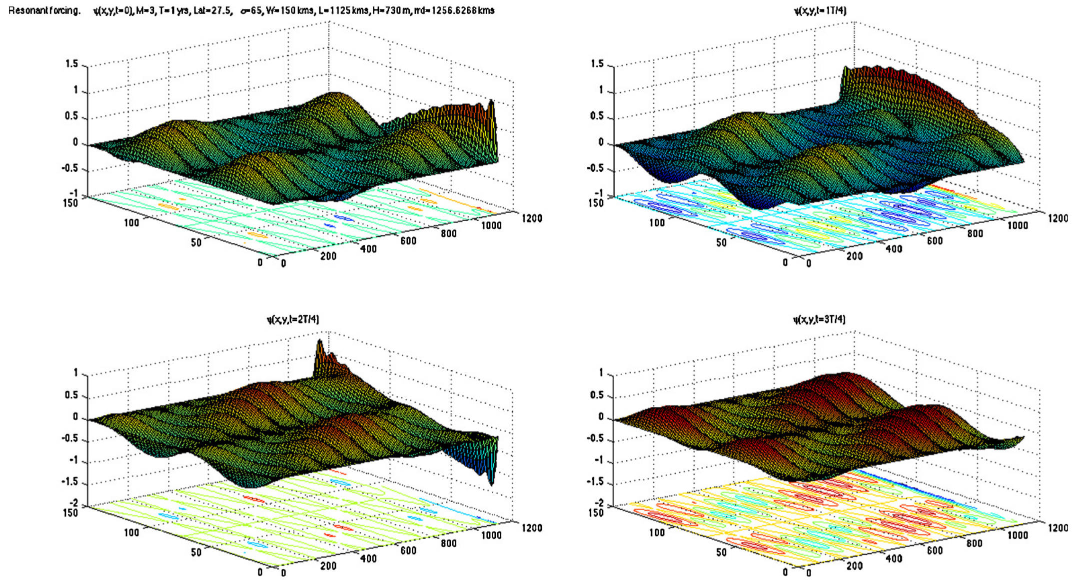

Fig. 6. $\psi(x, y, t)$ at $t = 0, T/4, T/2, 3T/4$ for a resonant mode $M = 3$ (evanescent). Parameters of Gulf of California: reference latitude = 27.5° N, $\alpha = 65°$, $W = 150$ km, $L = 1125$ km, $H = 730$ m $\rightarrow r_d = 1257$ km. $T = 1$ yr.

respectively. In both plots one can see the Gibbs effect at the end points of the gulf's mouth when the value of ψ there is not zero, i.e. the solution as a sum of gulf modes that vanish at $y = 0$, W can never achieve a non-zero value there and an overshooting is evident; in the interior the solution expansion (the sum) matches perfectly the boundary condition at the mouth. The solution for the oscillatory $M = 1$ resonant mode exhibits motion in the whole gulf with the same amplitude of the forcing, whereas in the evanescent $M = 3$ resonant mode the motion in the gulf's interior has an amplitude smaller than that of the forcing. In fact if one increases M, the motion is confined to be very near the mouth (the motion in the interior of the gulf is insignificant).

The URL site https://drive.google.com/drive/folders/0B7EmwKPi9NK6THo2M3h6TjZ1Xzg contains several animations (movies) of the solution for different cases; the reader is invited to consult them. In these movies the westward propagation is evident. Even in the case of a zonal gulf with a western mouth (i.e. $\alpha = 180°$) that is being forced there, one can see how the solution has a clear westward propagation towards the mouth!

3.2. Non-resonant forcing

In the non-resonant case (11) is valid and the mode amplitudes are:

$$a_m = \frac{2Fm\pi e^{i\beta L\cos\alpha/(2\omega)}\left\{1 - e^{i[\beta\sin\alpha/(2\omega)+\eta]W}(-1)^m\right\}}{W^2\sin(\delta_m L)\left\{m^2\pi^2/W^2 - \left[\beta\sin\alpha/(2\omega) + \eta\right]^2\right\}}. \tag{18}$$

It is worth noting that in this non-resonant forcing case, all modes get excited. The higher the mode number is, the less it contributes to the solution, since a_m is $O(1/m)$ or $O([m\sinh(m)]^{-1})$. However, the main contribution to the sum of the solution could come from those amplitudes near resonance. That is, for given ω and η, a_m would attain maximum values when m equals the nearest integer to the value $\pm[\beta\sin\alpha/(2\omega) + \eta]W/\pi$.

3.3. Special cases: prescribing η or the y dependence of the forcing

The previous analysis is general in the sense that the Fourier component of the forcing is not specified *a priori*, i.e. ω and η are free parameters, which are only linked in the resonant case through (13). To study special cases, one could prescribe the y dependence of the forcing or η from the beginning at the cost of loosing one degree of freedom in the forcing. Next, we discuss some special cases.

3.3.1. $\eta = 0$

There is no y dependence on the forcing, which physically means that the water goes up and down at the mouth with no flow through it. The resonance condition (13) is fulfilled for a couple of very particular frequencies that satisfy $\beta\sin\alpha/(2\omega) = \pm M\pi/W$, which is possible only in a non-zonal gulf.

If the gulf is zonally oriented ($\sin\alpha = 0$), then from (18) we see that only the odd modes are excited, i.e. $a_m = 0$ for m even. This is the well known result that only the odd modes contribute when expanding a constant function in terms of the complete set $\{\sin(my)|m = 1, 2, 3, \ldots\}$ in $[0, \pi]$.

An animation of the solution for the non-resonant case with the parameters resembling the Gulf of California is contained in the file NRFT1FconstGCH0.4.avi in the web site mentioned above.

3.3.2. $\eta = -i/r_d$ and F is multiplied by the constant $\exp(-W/r_d)$

The y-dependence of the forcing is $\exp[(y - W)/r_d]$ so the flow at the mouth has an exponential profile. There is a net transport through it, with more fluid entering (or leaving) the gulf on the $y = W$ side wall (northernmost for $\alpha < \pi/2$) than on the $y = 0$ side wall. In this case the amplitudes are

$$a_m = \frac{2Fm\pi e^{i\beta L\cos\alpha/(2\omega)}}{W^2\sin(\delta_m L)}\left[e^{-W/r_d} - e^{i\beta W\sin\alpha/(2\omega)}(-1)^m\right] \times \left[m^2\pi^2/W^2 + 1/r_d^2 - \beta^2\sin^2\alpha/(4\omega^2) - i\beta\sin\alpha/(\omega r_d)\right]$$
$$\times\left\{\left[m^2\pi^2/W^2 + 1/r_d^2 - \beta^2\sin^2\alpha/(4\omega^2)\right]^2 + \beta^2\sin^2\alpha/(\omega^2 r_d^2)\right\}^{-1}. \tag{19}$$

Note that the denominator can never vanish. This could be anticipated since η must be real to satisfy the resonance condition. In other words, there cannot be resonance because the y-dependence of the forcing is different from that of a gulf mode.

If the gulf is zonal, the amplitudes are

$$a_m = \frac{2Fm\pi e^{\pm i\beta L/(2\omega)}\left[e^{-W/r_d} - (-1)^m\right]}{W^2\sin(\delta_m L)\left(m^2\pi^2/W^2 + 1/r_d^2\right)}, \tag{20}$$

where the plus sign is for $\alpha = 0$ and the minus sign is for $\alpha = 180^\circ$, which corresponds to an eastern and western gulf's mouth, respectively.

An animation of the solution for the non-resonant case with parameters resembling the Gulf of California is contained in the file NRFT1FKelGCH0.4.avi at the web site mentioned above. Interestingly enough, the amplitude of ψ in the interior can reach 4–5 times that of the forcing at the mouth because several of the first modes that contribute to the solution interfere constructively.

3.3.3. $\psi = F\sin(M_f\pi y/W)e^{-i\omega t}$ at $x = L$, F = constant, $M_f > 0$ integer

To avoid the Gibbs effect, we now apply a forcing at the mouth that vanishes at $y = 0$, W at all times. This forcing could be recast in the form $Fe^{i(\eta y - \omega t)}$ by writing $\sin(M_f\pi y/W)$ in terms of exponentials so it would be the sum of two Fourier components with η given. However, we prefer the expression without the exponentials because the analysis is more clear in this case.

With this forcing, there is no net transport of mass into or out of the gulf at the mouth, no matter if M_f is even or odd because the mass transport is $T = \int_0^W u\,dy = -\int_0^W \partial_y\psi\,dy = -\psi|_0^W = 0$.

In the non-resonant case the amplitudes are:

$$a_m = \frac{2FM_f m\pi^2 i\beta\sin\alpha e^{i\beta L\cos\alpha/(2\omega)}\left[e^{i\beta W\sin\alpha/(2\omega)}(-1)^{M_f+m} - 1\right]}{W^3\omega\sin(\delta_m L)\left\{\left[(m^2 + M_f^2)\pi^2/W^2 - \beta^2\sin^2\alpha/(4\omega^2)\right]^2 - 4M_f^2 m^2\pi^4/W^4\right\}}. \tag{21}$$

In principle all modes get excited, even though we are forcing with the y-structure of one of the modes ($m = M_f$). The term responsible for this multi-mode excitation is $\exp[i\beta y\sin\alpha/(2\omega)]$ in the integral for the mode amplitudes, which is the non-zonality of the gulf.

An animation of the solution for $M_f = 2$ and for the non-resonant case with parameters resembling the Gulf of California is in the file NRFT1Fsin2GCH0.4.avi at the web site mentioned above. One can observe the absence of the Gibbs effect at the mouth.

Resonance would occur if for some $m = M$, $(M^2 + M_f^2)\pi^2/W^2 - \beta^2\sin^2\alpha/(4\omega^2) = \pm 2M_f M\pi^2/W^2$ and the frequency is calculated from $|M \mp M_f|\pi/W = \beta|\sin\alpha|/(2\omega)$. The amplitude for the resonant mode is $a_M = Fe^{i\beta L\cos\alpha/(2\omega)}/\sin(\delta_M L)$ and for $m \neq M$ they are

$$a_m = \frac{4FM_f mi\,\mathrm{sign}(\sin\alpha)\,|M \mp M_f|\,e^{i\beta L\cos\alpha/(2\omega)}\left[(-1)^{M+M_f+m} - 1\right]}{\pi\sin(\delta_m L)\left\{\left[m^2 + M_f^2 - (M \mp M_f)^2\right]^2 - 4M_f^2 m^2\right\}}. \tag{22}$$

If the gulf is zonal, we are forcing with exactly the y-structure of one of the modes, i.e. we have resonance and it is $a_m = 0$ for $m \neq M_f$ and $a_{M_f} = Fe^{\pm i\beta L/(2\omega)}/\sin(\delta_{M_f}L)$, i.e. only the mode $m = M_f$ of the forcing gets excited. The plus (minus) sign is for $\alpha = 0$ ($\alpha = 180^\circ$), an eastern (western) gulf's mouth.

In Fig. 7 we show two contour maps of the resonant period T of the forcing obtained from $\beta|\sin\alpha|/(2\omega) = M\pi/W$ as a function of the gulf orientation α and width W, for $M = 2$ and $M = 1$. This applies to the special cases 1 and 3; for example

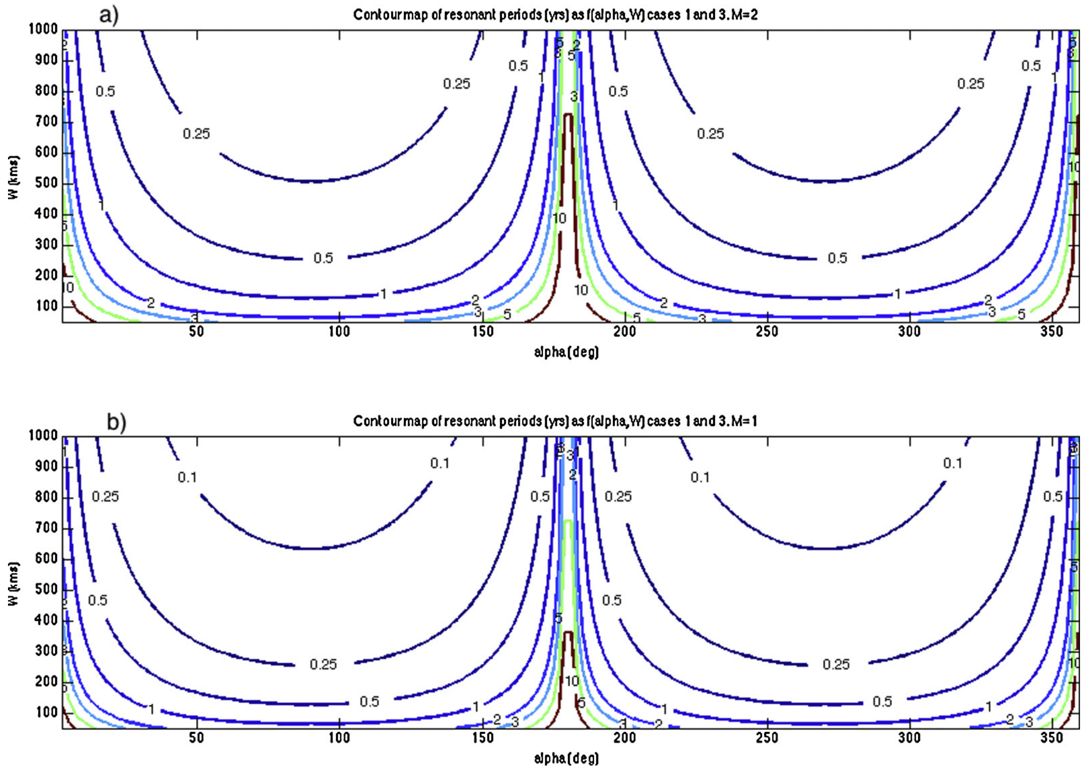

Fig. 7. Contour maps of the resonant period T (in years) of the forcing in the cases $\eta = 0$ and $\sin(M_f \pi y/W)$ as a function of the gulf orientation α (in °) and width W (in km) and for (a) $M = 2$; (b) $M = 1$. Reference latitude = 30° N.

$M = 2$ serves in case 3 for any M and M_f such that $|M \mp M_f| = 2$, etc. For most orientations and widths of the gulf between 50 and 1000 km the period of the resonant mode could range between 0.1 and 2 years or so. It is interesting to note that for gulf orientations between 40° and 130° and between 220° and 310° and for gulf widths between 150 and 200 km, the resonant mode is around the annual frequency for $M = 2$ (for $M = 1$ the gulf width should be less than 100 km). Two gulfs that come to mind having orientations and widths within these ranges of values are the Gulf of California and the Adriatic Sea. It is only near the values of $\alpha = 0°$, 180°, i.e. for nearly zonally oriented gulfs, that the periods of the resonant modes are larger than 2 years (more than 10 years for narrow gulfs less than 100 km wide). This is true regardless of the size of the Rossby radius of deformation or the gulf's length. If the resonant mode number M is increased, then the range of the resonant periods moves towards larger values.

4. Discussion and conclusions

In this note we find free and forced solutions of the QG linear potential vorticity equation in an idealised geometry: a rectangular gulf with a flat bottom whose orientation on the β-plane is arbitrary.

The free solutions are the Rossby normal modes in the gulf. A normal mode is constructed by reflecting a Rossby mode in a channel at the head of the gulf. So it can be viewed as the superposition of two Rossby channel modes: an incident and a reflected mode with respect to the wall at the head. Or it is also the superposition of four Rossby waves in an otherwise unbounded ocean arranged in such a way as to satisfy the boundary conditions of no normal flow at the three solid walls of the gulf. We show this construction geometrically, explaining that the two channel modes can have wavenumbers along the gulf axis that are either real and different, equal or complex conjugates. Depending on this, the gulf mode is oscillatory, grows linear or grows exponentially in x, the coordinate along the axis of the gulf, respectively. When the wavenumbers are complex conjugates, the slowness curve ω = constant in wavenumber space is a hyperbola (instead of the familiar circle for real wavenumbers).

The evanescent gulf modes that grow exponentially in x have values of ψ near the mouth that are several orders of magnitude larger than in the interior. If one puts a reasonable amplitude [say in terms of the horizontal velocity of O(1 m/s)], then the whole interior is motionless and the motion occurs only very near the mouth. On the contrary, the values of the oscillatory modes are of the same order everywhere in the domain of the motion. With this in mind it makes sense to study what the response of the gulf is (in terms of these free modes) when it is being forced at the mouth. To this end, we considered a single Fourier component of the forcing at the mouth $x = L$ of the form $\psi = Fe^{i(\eta y - \omega t)}$, i.e. we forced the gulf at a *fixed* frequency ω and with a wavenumber normal to the gulf axis η. It is physically an interesting problem and also analytically tractable. Note that this choice of forcing is not restrictive because having a linear problem, more general forcing functions can be considered by adding many Fourier components and the solutions for each component summed to get the total solution.

The forced solution is in general an infinite sum of gulf modes (all having the same frequency of the forcing), necessary to match the imposed forcing function at the mouth. The form of the mode amplitudes of the solution immediately calls to distinguish two cases: (a) resonant forcing, in which we force with the frequency ω and wavenumber component normal to the gulf's axis η exactly that of a gulf's mode, and thus frequency and wavenumber are linked through the resonance condition (13); and (b) non-resonant forcing, in which ω and η are not linked at all. The resonant forcing is physically important: out of all the modes that are excited with the imposed forcing, the dominant mode, i.e. the one that exhibits the largest response, is the resonant mode. For the given ω and mode number M, there are two wavenumbers ($\eta_{\pm}$) that are resonant (for given α and W). Or for the given η and M there are two resonant frequencies. It is also worth noting that all mode amplitudes a_m have $\sin(\delta_m L)$ in the denominator, which in the case of the evanescent modes keeps the term $\sin(\delta_m x)/\sin(\delta_m L)$ below 1 in the flow domain $x \leq L$; this is not the case for the free evanescent modes.

The solution in the form of an expansion of gulf modes exhibits the Gibbs effect. Because all gulf modes vanish at $y = 0, W$, if the imposed forcing function at the mouth does not vanish there, the expansion cannot match the forcing and overshoots it. In general, except for the third of the special cases, the forcing (resonant or not) does not vanish at the end points of the mouth.

A reviewer questioned whether the applied forcing at the mouth of the gulf includes the common tidal forcing as a contribution in Physical Oceanography. In principle, we may put any frequency in the analytical solution and in fact we calculated the solution for an $M = 1$ resonant forcing with a period of 1 day (a diurnal tide). With such frequency, all modes are evanescent and the response is confined to only a few km from the mouth, the rest of the gulf is motionless. Definitively, to consider tidal forcing other dynamics should be used, like the SWE that have the two gravitational modes.

Three special cases of forcing functions at the mouth were studied: (1) $\eta = 0$, which means that the water goes up and down at the gulf's mouth and there is no fluid entering or leaving the gulf; (2) $\eta = -i/r_d$ with the constant F redefined so that the y-dependence of the forcing is $\exp\left[(y - W)/r_d\right]$, which is a flow with an exponential profile resembling a Kelvin wave that has a net transport through it (but zero transport over one period); and (3) a flow with a sinusoidal profile $\sim \sin(M_f \pi y/W)$, where M_f is a positive integer, whose net transport is zero at all times.

Resonance occurs in cases (1) and (3) but only for very specific frequencies. In fact for case (1) resonance is possible only in non-zonal gulfs. Resonance is not possible in case (2) and the dominant modes will be simply the first modes, since $a_m \sim O(1/m)$ for the oscillatory modes and $a_m \sim O(1/[m \sinh(m)]$ for the rest of the evanescent modes. Therefore, the QG response in a gulf (given in terms of Rossby gulf modes) for a Kelvin wave type of forcing is never resonant. In case 3, the applied forcing vanishes at both ends of the gulf's mouth and the solution expansion does not exhibit the Gibbs effect, which implies that ψ and the velocity field are continuous everywhere; also if the gulf is zonal, the forcing is resonant and only the forcing mode $m = M_f$ gets excited.

As a potential application of our results to the physical oceanography of gulfs in the world's oceans, the $M = 1$ resonant period for the forcing in cases $\eta = 0$ and $\sim \sin(M_f \pi y/W)$ with M_f such that $|M \mp M_f| = 1$ is: 0.45 years for the Gulf of California (reference latitude = 27.5° N, $\alpha = 65°$, $W = 150$ km); 0.77 years for the Adriatic Sea (reference latitude = 42.5° N, $\alpha = 40°$, $W = 150$ km) and 0.24 years for the Red Sea (reference latitude = 20.1° N, $\alpha = 59.4°$, $W = 280$ km). Since T is linear in M, the periods for larger M are just multiples of these $M = 1$ periods. Of course, keep in mind that our gulf is idealised and that bottom topography, irregular coastlines, and so on might change these values.

For the forcing functions that we have studied, there is a qualitative difference in the response if the gulf is zonally oriented or not. In general, less modes get excited if the gulf is zonally oriented. This comes from the fact that (at least for the forcing functions that have fluid entering or leaving the gulf) in the non-zonal gulf we are forcing water to cross lines of ambient potential vorticity, thereby activating the β-effect responsible for planetary wave motion and thus exciting many more modes.

In spite of our gulf being idealised, the analytical results presented here could provide a dynamical basis to help explain observations. And for sure, the analytical solutions could be a very useful tool to test numerical models of gulfs. Beyond these benefits, we have contributed to the advancement of knowledge in GFD.

Acknowledgments

This work was supported by CICESE under internal projects 621-111 and 621-137 and through a sabbatical scholarship awarded by the National Council for Science and Technology of Mexico (CONACYT) under application number 265306. This paper was completed while the author was spending a sabbatical year at the Abdus Salam International Centre for Theoretical Physics (ICTP). I am grateful to Filippo Giorgi of ICTP for his invitation and support. Many thanks to Ismael Villanueva who was very helpful with his technical assistance in processing the manuscript with Latex.

References

Beier, E., 1997. A numerical investigation of the annual variability in the Gulf of California. J. Phys. Oceanogr. 27, 615–632.
García, R., Graef, F., 1998. The nonlinear self-interaction of a baroclinic Rossby mode in a channel and a gulf. Dyn. Atmos. Oceans 28, 139–155.
Graef, F., Magaard, L., 1994. Reflection of nonlinear baroclinic Rossby waves and the driving of secondary mean flows. J. Phys. Oceanogr. 24, 1867–1894.

Graef, F., Müller, P., 1996. Uniqueness of solutions and conservation laws for the quasigeostrophic model. Dyn. Atmos. Oceans 25, 109–132.
Hamilton, P., 2009. Topographic Rossby waves in the Gulf of Mexico. Prog. Oceanogr. 82, 1–31.
Harlander, U., Ridderinkhof, H., Schouten, M.W., De Ruijter, W.P.M., 2009. Longterm observations of transport, eddies, and Rossby waves in the Mozambique Channel. J. Geophys. Res. Oceans 114, C02003, http://dx.doi.org/10.1029/2008JC004846.
Hurlburt, H.E., Thompson, J.D., 1982. The dynamics of the Loop Current and shed eddies in a numerical model of the Gulf of Mexico. Elsevier Oceanogr. Ser. 34, 243–297.
Lluch-Cota, S.E., Pars-Sierra, A., Magaa-Rueda, V.O., Arregun-Snchez, F., Bazzino, G., Herrera-Cervantes, H., Lluch-Belda, D., 2010. Changing climate in the Gulf of California. Prog. Oceanogr. 87, 114–126.
Oey, L.-Y., Lee, H.-C., 2002. Deep eddy energy and topographic Rossby waves in the Gulf of Mexico. J. Phys. Oceanogr. 32, 3499–3527.
Pasarić, M., Pasarić, Z., Orlić, M., 2000. Response of the Adriatic sea level to the air pressure and wind forcing at low frequencies (0.01–0.1 cpd). J. Geophys. Res. Oceans 105, 11423–11439.
Pinardi, N., Milliff, R., 1990. A note on consistent quasigeostrophic boundary conditions in partially open, simply and multiply connected domains. Dyn. Atmos. Oceans 14, 65–76.
Ripa, P., 1990. Seasonal circulation in the Gulf of California. Ann. Geophys. 8, 559–564.
Ripa, P., 1997. Toward a physical explanation of the seasonal dynamics and thermodynamics of the Gulf of California. J. Phys. Oceanogr. 27, 597–614.
Shu, Y., Xue, H., Wang, D., Chai, F., Xie, Q., Cai, S., Chen, R., Chen, J., Li, J., He, Y., 2016. Persistent and energetic bottom-trapped topographic Rossby waves observed in the southern South China Sea. Sci. Rep. 6, 24338, http://dx.doi.org/10.1038/srep24338.
Wu, X., Xie, Q., He, Z., Wang, D., 2008. Free and forced Rossby waves in the western South China Sea inferred from Jason-1 satellite altimetry data. Sensors 8, 3633–3642.
Yang, H.J., Liu, Q.Y., 2003. Forced Rossby wave in the northern South China Sea. Deep Sea Res. I 50, 917–926.